AXIOMATIC FRAMEWORK FOR THE BGG CATEGORY $\mathcal{O}$

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ABSTRACT. The main goal of this paper is to show that a wide variety of infinite-dimensional algebras all share a common structure, including a triangular decomposition and a theory of weights. This structure allows us to define and study the BGG Category $\mathcal{O}$, generalizing previous definitions of it.

Having presented our axiomatic framework, we present sufficient conditions that guarantee finite length, enough projectives, and a block decomposition into highest weight categories. The framework is strictly more general than the usual theory of $\mathcal{O}$; this is needed to accommodate (quantized or higher rank) infinitesimal Hecke algebras, in addition to semisimple Lie algebras and their quantum groups.

We then present numerous examples, two families of which are studied in detail. These are quantum groups defined using not necessarily the root or weight lattices (for these, we study the center and central characters), and infinitesimal Hecke algebras.

1. Introduction

This paper is motivated by the study of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [BGG1] associated with a complex semisimple Lie algebra $\mathfrak{g}$. The definition of $\mathcal{O}$ depends on the fact that $\mathfrak{U}\mathfrak{g}$ has a triangular decomposition. This category has been studied quite intensively; to name but a few references, see [AnSt, BGG2, Kon, Maz, Soe]. (Also see the very recent book [H2].) One important property of the category $\mathcal{O}$ is that its blocks are highest weight categories in the sense of [CPS].

Our main aim in this paper, is to simultaneously generalize this setup in various different ways. First, many well-known algebras in representation theory possess a triangular decomposition similar to $\mathfrak{U}\mathfrak{g}$ - for example, symmetrizable Kac-Moody Lie algebras [Kac] and their quantum analogs [Ja, Jo], contragredient Lie algebras [KK], the (centerless) Virasoro algebra [FF], and extended (centerless) Heisenberg algebras [MP, §1.4].

Thus, it is natural to define an analogue of the category $\mathcal{O}$ for each of these algebras. However, one crucial component of the theory for $\mathfrak{U}\mathfrak{g}$ involved

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using central characters - via a finiteness condition that we call (S4); this works because the center of $\mathfrak{U}\mathfrak{g}$ is large enough.

In general, however, this is false: there are algebras with triangular decomposition, whose center is trivial. Thus, one of our main results is to propose a strictly weaker condition (which we call (S3), and which holds for $\mathfrak{U}\mathfrak{g}$ because of the analysis involving central characters - i.e., (S4) - and) which also implies a block decomposition into highest weight categories.

We also generalize the notion of a triangular decomposition, beyond what was carried out in the literature (see [RCW],[MP]). Thus, we are able to include semisimple Lie algebras $\mathfrak{U}\mathfrak{g} = \mathfrak{U}\mathfrak{n}_- \otimes \mathfrak{U}\mathfrak{h} \otimes \mathfrak{U}\mathfrak{n}_+$ and their quantum groups $U_q(\mathfrak{g}) = U_q(\mathfrak{n}_-) \otimes U_q^0 \otimes U_q(\mathfrak{n}_+)$ in the same framework (as well as the other examples above). This notion is a very natural one, since all the above examples, and many more (some of which we describe in this paper), fit into this theory.

There are similarities between our framework and that of [AnSch], in that Hopf algebras, weight spaces, and quantum groups are involved. However, this construction is significantly different as well: the algebras here are neither finite-dimensional, nor do they need to be Hopf algebras (and a priori, we also do not impose restrictions on the ground field).

It is only natural now, to ask for the right notion of the BGG category $\mathcal{O}$ in this setting. In all the examples above, various theories have been proposed; our version includes many of them as special cases - and in particular, the original setup in [BGG1].

There is another reason why this notion is the right one: we were able to use it to study the infinitesimal Hecke algebra of $\mathfrak{sl}_2$ (and $\mathbb{C}^2$) in [Kh1] and (with Tikaradze,) in [KT] - as well as its quantized analogue (with Gan) in [GK]. In these papers, we obtain nontrivial representations for these algebras in the setting of our proposed category $\mathcal{O}$.

Finally - there is another strict weakening, of the axioms for $\mathcal{O}$ that have been used in the literature thus far. In all the examples mentioned above, if we denote the triangular decomposition as $A = B_- \otimes H \otimes B_+$, then one requires the weights of $B_\pm$ to lie in positive and negative cones with respect to the entire algebra H (or $\mathfrak{h}$). However, we only require this condition to hold with respect to some Lie subalgebra $\mathfrak{h}_0$ of $\mathfrak{h}$ (or more precisely, a subalgebra $H_0 \subset H = \mathfrak{U}\mathfrak{h}$).

The good news is that the conditions (S3) and (S4) referred to above, can be generalized to this weaker setup - and even now, lead to a decomposition of $\mathcal{O}$ into highest weight categories. This allows us to consider certain higher rank infinitesimal Hecke algebras, and we conclude by establishing some results for them.

This paper is organized as follows. In §2, we present the framework encompassing a wide variety of algebras. Section 3 discusses the special case when H is a Hopf algebra, and ends by characterizing such algebras.

Next, we introduce Verma modules and other key concepts in §4, and state a few results in §5 that yield information about $\mathcal{O}$.

The second half is devoted to examples. Section 6 contains a miscellany of items, and §7 presents many examples of Lie algebras with "regular triangular decomposition". We next define and study (in §8) a family of "quantum groups" associated to any semisimple Lie algebra, generalizing what to use instead of the coroot lattice. We next provide more examples in §9, and finally in §10, we study the motivating example for this "weaker" axiomatic setup: infinitesimal Hecke algebras.

2. The main definition

We work throughout over a ground field k. All tensor products below are over k, unless otherwise specified. We define $\mathbb{Z}_{\geqslant 0} = \mathbb{N} \cup \{0\}$. Given $S \subset \mathbb{Z}$ and a finite subset Δ of an abelian group $\mathcal{P}_0$, the symbols $(\pm)S\Delta$ stand for $\{(\pm)\sum_{\alpha \in \Delta} n_\alpha \alpha : n_\alpha \in S\ \forall \alpha\} \subset \mathcal{P}_0$. We will often abuse notation and claim that two modules or functors are equal, when they are isomorphic (double duals, for instance).

Definition 2.1. Given a k-algebra A, the set of *weights* is $\mathrm{Hom}_{k-alg}(A, k)$. Given a weight λ and an A-module M, the λ-*weight space* of M is $M_\lambda := \{m \in M : am = \lambda(a)m\ \forall a \in A\}$.

All our results on the category $\mathcal{O}$ are proved in a general setup that we now mention. On the other hand, all the examples found in the literature fall under a slightly more restricted framework (described in Section 3). Thus, for "all practical purposes", the reader should skip this section completely, and go ahead directly to Definition 3.6.

Definition 2.2. An associative k-algebra A, together with the following data, is called a *regular triangular algebra* (denoted also by *RTA*).

(RTA1) There exist associative unital k-subalgebras $B_\pm, H$ of A, such that the multiplication map : $B_- \otimes_k H \otimes_k B_+ \to A$ is a vector space isomorphism (the *triangular decomposition*).

(RTA2) There is an algebra map $\mathrm{ad} \in \mathrm{Hom}_{k-alg}(H, \mathrm{End}_k(A))$, such that for all $h \in H$, $\mathrm{ad}\, h$ preserves each of $H, B_\pm$ (identifying them with their respective images in A). Moreover, $H \otimes B_\pm$ are k-subalgebras of A.

(RTA3) There exists a free action $*$ of a group $\mathcal{P}$ on $G := \mathrm{Hom}_{k-alg}(H, k)$, as well as a distinguished element $0_G = 0_\mathcal{P} * 0_G \in G$ such that $H = H_{0_G}$ as an $\mathrm{ad}\, H$-module.

(RTA4) There exists a subalgebra H_0 of H, and a free abelian group $\mathcal{P}_0$ of finite rank, such that
  - (a) $\mathcal{P}_0$ acts freely on $G_0 := \mathrm{Hom}_{k-alg}(H_0, k)$ (call this action $*$ as well), and
  - (b) the "restriction" map $\pi : G \to G_0$ sends $\mathcal{P} * 0_G$ onto $\mathcal{P}_0 * \pi(0_G)$, and intertwines the actions, i.e., $* \circ (\pi \times \pi) = \pi \circ *$.

For the remaining axioms, we need some notation. Fix a finite basis Δ of $\mathcal{P}_0$. For each $\theta \in \mathcal{P}$ and $\theta_0 \in \mathcal{P}_0 = \mathbb{Z}\Delta$, abuse notation and define $\theta = \theta * 0_G \in G$, $\theta_0 = \theta_0 * \pi(0_G) \in G_0$. (We will differentiate between $0 \in \mathcal{P}_0$ or G_0, and $0_G \in G$.) We call G (or $G_0, \mathcal{P}_0, \Delta$) the set of *weights* (or the *restricted weights, root lattice, simple roots* respectively).

Given $\lambda \in S \subset G$ and a module M over H (e.g., $M = (A, \mathrm{ad})$), define the *weight space* M_λ as above, and $M_S := \bigoplus_{\lambda \in S} M_\lambda$. Given $\theta_0 \in \mathbb{Z}\Delta$, define $M_{\theta_0} := M_{\pi^{-1}(\theta_0)}$.

(RTA5) It is possible to choose Δ, such that $B_\pm = \displaystyle\bigoplus_{\theta \in \mathcal{P} : \pi(\theta) \in \pm\mathbb{Z}_{\geqslant 0}\Delta} (B_\pm)_\theta$ (where A is an H-module via ad).

(RTA6) $(B_\pm)_0 = (B_\pm)_{\pi^{-1}(\pi(0_G))} = k$, and $\dim_k (B_\pm)_{\theta_0} < \infty \ \forall \theta_0 \in \pm\mathbb{Z}_{\geqslant 0}\Delta$ (we call this *regularity*).

(RTA7) The *property of weights* holds: for all A-modules M,

$$\begin{aligned} A_\theta \cdot A_{\theta'} &\subset A_{\theta * \theta'} \ \forall \theta, \theta' \in \mathcal{P}, \\ A_\theta \cdot M_\lambda &\subset M_{\theta * \lambda} \ \forall \theta \in \mathcal{P}, \lambda \in G. \end{aligned}$$

(RTA8) There exists an anti-involution i of A (i.e., $i^2|_A = \mathrm{id}|_A$) that acts as the identity on all of H, and takes A_θ to $A_{\theta^{-1}}$ for each $\theta \in \mathcal{P}$.

Definition 2.3. An RTA is *strict* if $H = H_0, G = G_0 \supset \mathcal{P} = \mathcal{P}_0$ (whence $\pi = \mathrm{id}|_G$).

We need this more general notion to study infinitesimal Hecke algebras later.

Example. This definition is quite technical; here is our motivating example - a complex semisimple Lie algebra $\mathfrak{g}$. Then $A = \mathfrak{Ug}$ is a strict RTA, ad is the standard adjoint action (of the Lie algebra, or the Hopf algebra), and $H = H_0 = \mathrm{Sym}\,\mathfrak{h}$, whence the set of weights is $G = G_0 = \mathfrak{h}^* \supset \mathcal{P} = \mathcal{P}_0 = \mathbb{Z}\Delta$ (the root lattice). Moreover, i is the composite of the Chevalley involution and the Hopf algebra antipode on $\mathfrak{Ug}$.

Remark 2.4.

(1) The last axiom implies that H is commutative; also, we have subalgebras (actually, augmentation ideals) in $B\pm$, defined (respectively) as: $N_\pm := \bigoplus_{\theta \neq 0_G} (B_\pm)_\theta$. Moreover, $A = \bigoplus_{\theta \in \mathcal{P}} A_\theta = \bigoplus_{\theta_0 \in \mathbb{Z}\Delta} A_{\theta_0}$.
(2) If $(G_0, *)$ is a k-vector space under addition, then we also require k to have characteristic zero, since $\mathbb{Z}\Delta$ would otherwise have torsion. Thus, $\mathrm{char}(k) = 0$ for Lie algebras, but not for quantum groups.
(3) We give many well-known and widely studied examples below, and in these cases, $H = H_0$ is a commutative and cocommutative Hopf algebra generated by its primitive and group-like elements, and ad its adjoint action. In such a setup, one can prove some of the assumptions mentioned above, as we now see in §3.

3. Hopf algebras satisfy some of the assumptions

We now analyze the case when H is a Hopf algebra, and H_0 a Hopf subalgebra. We use standard results on Hopf algebras for this (see [Abe], or [ES, Lecture 8]). We adopt the following notation henceforth:

Notation. Let H be a Hopf algebra (not necessarily commutative) over a field k, and for every Hopf algebra, let m (or $\Delta, \eta, \varepsilon, S$) denote henceforth, the multiplication (or comultiplication, unit, counit, antipode respectively).

Now define $G := \operatorname{Hom}_{k-alg}(H, k) \subset H^*$, the set of "grouplike elements" in H^*. Also define *convolution* on G, via $\langle \mu * \lambda, h\rangle := \langle \mu \otimes \lambda, \Delta(h)\rangle = \sum \langle \mu, h_{(1)}\rangle\langle \lambda, h_{(2)}\rangle$, using Sweedler notation. Then ([Abe, Theorem 2.1.5]) $(G, *)$ is a group, with unit ε, and inverse given by $\lambda \mapsto \lambda \circ S$ in G.

3.1. Preliminaries.

We now mention a *property of weights*, for a more general class of adjoint actions.

Lemma 3.1 (Weight Lemma)**.** *(H, G as above.) Let A be a k-algebra containing H, with unit $\eta_A(1) = 1_A = 1_H = \eta_H(1)$. Fix $f, g \in \operatorname{End}_k H$, with $m(f \otimes g)(\Delta(h)) = \eta(\varepsilon(h))\ \forall h \in H$, and define $\operatorname{ad}_A \in \operatorname{Hom}_k(H, \operatorname{End}_k A)$ by*

$$\operatorname{ad}_A h(a) = \sum g(h_{(1)}) a f(h_{(2)})$$

Given an A-module M and $\zeta \in \operatorname{Hom}_A(M \otimes H, M)$ (where $a \cdot (m \otimes h) = (am) \otimes h$), define an "adjoint action" $\operatorname{ad}_\zeta \in \operatorname{Hom}_k(H, \operatorname{End}_k M)$ now, by

$$\operatorname{ad}_\zeta h(m) = \zeta\left(\sum g(h_{(1)}) m \otimes h_{(2)} \right)$$

1. *Let A_μ and M_λ denote the* weight spaces *with respect to these actions, for $\mu, \lambda \in G$. Then $A_\mu M_\lambda \subset M_{\mu * \lambda}$.*
2. *Further assume that g is multiplicative, and ζ the action map for a right H^{op}-module structure on M. Then ad_ζ is an algebra map $: H \to \operatorname{End}_k M$.*

In the above result, H^{op} stands for H with the opposite multiplication. Also note that the usual adjoint action of H arises when we set $f = \operatorname{id}|_H, g = S$.

Proof of the Weight Lemma. Neither part is hard to show; we only show the first part here. We start with

$$\langle \mu * \lambda, h\rangle a_\mu m_\lambda = \sum (\langle \mu, h_{(1)}\rangle a_\mu) \cdot (\langle \lambda, h_{(2)}\rangle m_\lambda).$$

Now, $\langle \mu, h_{(1)}\rangle a_\mu = \operatorname{ad}_A h_{(1)}(a_\mu)$ by definition of weight spaces. Similarly, $\langle \lambda, h_{(2)}\rangle m_\lambda = \operatorname{ad}_\zeta h_{(2)}(m_\lambda)$. Hence we expand these out to get

$$\langle \mu * \lambda, h\rangle a_\mu m_\lambda = \sum g(h_{(1)}) a_\mu f(h_{(2)}) g(h_{(3)}) \zeta(m_\lambda \otimes h_{(4)}) \tag{3.2}$$

since ζ is an A-module map. Now compute:

$$\begin{aligned} & \sum g(h_{(1)}) \otimes f(h_{(2)}) g(h_{(3)}) \otimes h_{(4)} = (g \otimes m(f \otimes g) \otimes 1)(\Delta^3(h)) \\ = \quad & (g \otimes (\eta \circ \varepsilon) \otimes 1)(\Delta^2(h)) = (g \otimes \varepsilon \otimes 1)\Delta^2(h) \\ = \quad & (g \otimes 1 \otimes 1)((1 \otimes \varepsilon)\Delta \otimes 1)\Delta(h) = (g \otimes 1)\Delta(h). \end{aligned}$$

Applying ζ, we now rewrite equation (3.2) as

$$\langle \mu * \lambda, h \rangle a_\mu m_\lambda = \sum g(h_{(1)}) a_\mu \zeta(m_\lambda \otimes h_{(2)}),$$

and this equals $\operatorname{ad}_\zeta h(a_\mu m_\lambda)$, since ζ is a map of left A-modules. $\square$

We can now state and prove the promised result, that simplifies the definition of an RTA, when H and H_0 are both Hopf algebras.

Proposition 3.3. *Let* $\operatorname{ad} : H \to \operatorname{End}_k A$ *be the usual adjoint action, for* $A \supset H$ *as above.*

1. *Then* ad *is an algebra map* $: H \to \operatorname{End}_k A$.
2. *The property of weights holds for* A*, that is,* $A_\mu A_\lambda \subset A_{\mu * \lambda}$*, and* $A_\mu M_\lambda \subset M_{\mu * \lambda}$ *for every* $\mu, \lambda \in G$*, and* A*-module* M.
3. *The centralizer* $\mathfrak{Z}_A(H)$ *of* H *in* A *equals* A_ε*. Thus, the center* $\mathfrak{Z}(A)$ *is in* A_ε*, and* H *is commutative if and only if* $H = H_\varepsilon$.
4. *Suppose* i *is an anti-homomorphism on* A*. If* $i|_H = \operatorname{id}|_H$*, then for all* $\mu \in G$*,* $i(A_\mu) \subset A_{\mu^{-1}}$.

Remark 3.4. To show the last assertion, we need the following result (see [Abe, Theorem 2.1.4]):

Theorem 3.5. *Suppose* H *is a Hopf algebra over a field* k.

1. *The following are equivalent:*
   (a) $m^{op}(1 \otimes S)\Delta = m(S \otimes 1)\Delta^{op}$ *equals* $\eta \circ \varepsilon$ *on all of* H.
   (b) $m^{op}(S \otimes 1)\Delta = m(1 \otimes S)\Delta^{op}$ *equals* $\eta \circ \varepsilon$ *on all of* H.
   (c) $S^2 = \operatorname{id}$ *on* H.
2. *Thus, if* H *is commutative or cocommutative, then* $S^2 = \operatorname{id}$ *on* H.

Proof of Proposition 3.3.

1. Set $g = \operatorname{id}_H$, $f = S$, $M = A$, and $\zeta = m_A \circ (1 \otimes S)$, where m_A denotes the multiplication in A. Then $\operatorname{ad} = \operatorname{ad}_\zeta$. Moreover, (A, ζ) is now an (A, H^{op})-bimodule as required. Hence ad is an algebra map, by the Weight Lemma above.
2. That $A_\mu A_\lambda \subset A_{\mu * \lambda}$ follows from the Weight Lemma 3.1 above, using the values for f, g, M, ζ used in the previous part. Similarly, the condition for M follows by setting $f = S$, $g = \operatorname{id}_H$, and $\zeta = \operatorname{id}_M : M \to M$. More precisely, $\zeta = \operatorname{id}_M \otimes (\eta \circ \varepsilon)$, for then

$$\operatorname{ad}_\zeta h(m) = \sum h_{(1)} m \eta(\varepsilon(h_{(2)})) = \sum h_{(1)} \varepsilon(h_{(2)}) m = (\operatorname{id} \otimes \varepsilon)\Delta(h) \cdot m = hm$$

   (Note that ad_ζ is an algebra map here as well.)

3. This is from [Jo, Lemma 1.3.3].

(4) Since $i|_H = \mathrm{id}\,|_H$, hence H is commutative, so that $S^2 = \mathrm{id}$ on H (from above). Now, given the equation $\mathrm{ad}\, h(a_\mu) = \mu(h) a_\mu$, apply i and use the properties of Hopf algebras, to prove the result.

□

3.2. **The simpler (Hopf) setup.** We can now present the simplified version of an RTA.

Definition 3.6. A *Hopf regular triangular algebra* (or *HRTA* in short), is a k-algebra A, together with the following data.

(HRTA1) The multiplication map : $B_- \otimes_k H \otimes_k B_+ \to A$ is a vector space isomorphism, where $H, B_\pm$ are associative unital k-subalgebras of A, and H is, in addition, a commutative Hopf algebra.

(HRTA2) H contains a sub-Hopf algebra H_0 (with groups of weights G, G_0 respectively), and G_0 contains a free abelian group of finite rank $\mathcal{P}_0 = \mathbb{Z}\Delta$. Here, Δ is a basis of $\mathcal{P}_0$, chosen such that

$$B_\pm = \bigoplus_{\theta \in G : \pi(\theta) \in \pm \mathbb{Z}_{\geqslant 0} \Delta} (B_\pm)_\theta = \bigoplus_{\theta_0 \in \pm \mathbb{Z}_{\geqslant 0} \Delta} (B_\pm)_{\theta_0},$$

where $\pi : G \to G_0$ is the restriction (to H_0) map (and the summands are weight spaces under the usual adjoint actions). Each summand in the second sum is finite-dimensional, and $(B_\pm)_{0_G} = (B_\pm)_0 = k$.

(HRTA3) There exists an anti-involution i of A, such that $i|_H = \mathrm{id}\,|_H$.

Definition 3.7. A HRTA is *strict* if $H = H_0$ (whence $G = G_0 \supset \mathcal{P}_0 = \mathbb{Z}\Delta$).

The main result in this section is:

Theorem 3.8. *A k-algebra A is a (strict) HRTA if and only if A is a (strict) RTA and $H \supset H_0$ are Hopf algebras, with* ad *the usual adjoint action.*

We also mention that Hopf regular triangular algebras are examples of *algebras with triangular decomposition over H*, in the sense of [BaBe]; see [Kh2] for more details.

Proof. We can ignore strictness in this proof. Suppose A is an HRTA; we show that it is also an RTA. Since $B_\pm$ are direct sums of weight spaces, as is H ($H = H_\varepsilon$ by Proposition 3.3; define $0_G := \varepsilon$), (RTA1) and most of (RTA2) hold. Now define $\mathcal{P} := \pi^{-1}(\mathcal{P}_0) \subset G$. Since π is the restriction map, hence given $h_0 \in H_0$ and $\lambda, \lambda' \in G$,

$$\pi(\lambda * \lambda')(h_0) = (\lambda * \lambda')(h_0) = \sum \lambda((h_0)_{(1)}) \lambda'((h_0)_{(2)}) = (\pi(\lambda) * \pi(\lambda'))(h_0),$$

whence (RTA3) and (RTA4) hold too (since subgroups $\mathcal{P}, \mathcal{P}_0$ act freely on groups G, G_0 respectively). The axioms for an HRTA already contain (RTA5) and (RTA6), and Proposition 3.3 implies (RTA7) and (RTA8).

It remains to check that $H \otimes B_\pm$ are subalgebras of A. Although these are smash product algebras, H need not be cocommutative, so we use H-semisimplicity instead. Given $h \in H, \mu \in G$, and a weight vector $b \in (B_\pm)_\mu$,

$$hb = \sum \mu(h_{(1)}) b h_{(2)}, \qquad bh = \sum \mu^{-1}(h_{(1)}) h_{(2)} b$$

(where we use the first equation to show the second one). Now extend this to all $b \in B$, by linearity.

Conversely, suppose A is an RTA with $H \supset H_0$ Hopf algebras (and ad the usual adjoint action). Then define $\mathcal{P}_0 := \mathcal{P}_0 * 0 \subset G_0$. Now verify all the HRTA axioms (note that we must have $0_G = \varepsilon$). $\square$

3.3. **A recipe for producing central elements.** We conclude this section with the following result (and some of its consequences), whose special case is needed in Section 10 below. The proof is standard: for example, it coincides with the proof of the result that the *quadratic Casimir element* is central in $\mathfrak{Ug}$, for a semisimple Lie algebra $\mathfrak{g}$.

Proposition 3.9. *Suppose H is a Hopf algebra, and A an H-module algebra. If V, V' are finite-dimensional subspaces of A such that $V' \cong V^*$ as H-modules, then $\sum_i v_i v_i^* \in A_\varepsilon = \mathfrak{Z}_A(H)$, where ε is the counit in H, and v_i, v_i^* denote dual bases of V, V' respectively.*

Apart from being used to produce a central element (the Casimir) in $\mathfrak{Ug}$, this trick has been applied several times in [KT]. We generalize those attempts - and others - in the following corollary.

Corollary 3.10.

1. *Suppose H is a Hopf algebra, and M an H-module. If V, V' are finite-dimensional dual H-modules that occur as subspaces of* $\operatorname{Sym} M$*, then $\sum_i v_i v_i^*$ is central in $H \ltimes \operatorname{Sym}(M)$, where v_i, v_i^* run over dual bases of V, V' respectively.*
2. *Suppose $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{Z}(\mathfrak{g})$ is a complex reductive Lie algebra, and V a $\mathfrak{g}$-module, together with a $\mathfrak{g}$-module injection $\varphi : \mathfrak{g}_{ss} \to \operatorname{Sym} V$. Then $\sum_i \varphi(X_i)^2$ is central in $\mathfrak{U}(\mathfrak{g} \ltimes V)$, where $\{X_i\}$ is an orthonormal basis of (the semisimple part) $\mathfrak{g}_{ss}$ with respect to the Killing form.*
3. *Keeping the same notation, $\sum_i X_i \varphi(X_i)$ is central in the subalgebra generated by $\mathfrak{g}$ and $M = \varphi(\mathfrak{g}_{ss})$.*

Examples of this corollary abound: in [KT], it was used in producing central elements in $\mathfrak{U}(\mathfrak{gl}_n \ltimes (\mathbb{C}^n \oplus (\mathbb{C}^n)^*))$ and $\mathfrak{U}(\mathfrak{sp}_{2n} \ltimes \mathbb{C}^{2n})$ for all n. Here is another well-known example (see [Tak]): the *Takiff algebra* $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}$ always has the invariants $\mathfrak{h} := \mathfrak{Z}(\mathfrak{g})$ and $\varphi(\mathfrak{h})$ (this is easy to show), as well as the quadratic Casimir elements $\sum_i X_i \varphi(X_i)$ and $\sum_i \varphi(X_i)^2$ (set $M = \mathfrak{g}_{\mathrm{ad}}$ above).

4. Basic definitions, functoriality, Verma modules

First, we have the following easy result.

Lemma 4.1. *Every commutative Hopf k-algebra H is a strict Hopf RTA.*

Indeed, we can choose $B_\pm = k$, $(\mathcal{P} =)\mathcal{P}_0 = \{\varepsilon\}$ (so $\Delta = \emptyset$), and $i = \mathrm{id}$.

Next, given (strict) (Hopf) RTAs, we can construct more examples:

Proposition 4.2. *If A_i are (strict) (Hopf) RTAs for all i, then so is $A := \otimes_{i=1}^n A_i$.*

(The multiplication in A is coordinate-wise.) The proof is straightforward, but involves checking details; use Theorem 3.8 in the case of Hopf RTAs.

Next, we introduce a few concepts that help us analyze various examples from the literature, in the next part of this paper.

Definition 4.3. A is an RTA.

1. We define a *partial order* $\geq$ on G_0, by: $\lambda \geq \mu$ if and only if $\exists \theta_0 \in \mathbb{Z}_{\geqslant 0}\Delta$ such that $\theta_0 * \mu = \lambda$. Now define a partial order on G, via: $\lambda \geq \mu$ if $\pi(\lambda) > \pi(\mu)$ in G_0, or $\lambda = \mu$ in G.
2. The *BGG Category $\mathcal{O}$* (see [BGG1]) is the full subcategory of all finitely generated H-semisimple A-modules with finite-dimensional weight spaces, on which B_+ acts locally finitely.
3. Given $\lambda \in G$, the corresponding *Verma module* is $Z(\lambda) := A/A \cdot (N_+, \{h - \lambda(h) \cdot 1 : h \in H\})$, and $V(\lambda)$ is its unique simple quotient.
4. A *maximal vector* of weight λ in an A-module M, is $m \in M_\lambda \cap \ker N_+$.
5. For each $\lambda \in G$, define three sets:
   - $S^3(\lambda)$ is the equivalence closure of $\{\lambda\}$ in G, under the relation:
     $\mu \to \lambda$ if and only if $V(\mu)$ is a subquotient of $Z(\lambda)$.
   - $S^2(\lambda) := \{\pi(\mu) : \mu \in S^3(\lambda)\}$.
   - $S^1(\lambda) := \{\pi(\mu) : \mu \in S^3(\lambda), \mu \leq \lambda\}$.
6. We say that the algebra A satisfies *Condition (S1),(S2)*, or *(S3)* if the corresponding set $S^n(\lambda)$ is finite for each $\lambda \in G$.
7. The *block* $\mathcal{O}(\lambda)$ consists of all objects in $\mathcal{O}$, each of whose simple subquotients is in $\{V(\mu) : \mu \in S^3(\lambda)\}$.
8. The *Harish-Chandra projection* is $\xi : A = H \oplus (N_- \cdot A + A \cdot N_+) \twoheadrightarrow H$.

Remark 4.4. The S-sets are named like the "T"-properties of separation/Hausdorffness in point-set topology. (We relate (S1)–(S4) in Corollary 4.7 below.) Moreover, they should *not* be confused with the antipode map on H, since this map does not show up after the previous section of this paper (except once in Theorem 8.3).

Also see subsection 5.1 for further comments on the Conditions (S).

We now mention some basic results; the proofs are standard.

Proposition 4.5. *Suppose A is an RTA, and $\lambda \in G$.*

1. *$\geq$ is a partial order on G_0, G.*
2. *$Z(\lambda) \in \mathcal{O}$, and it is generated by its one-dimensional λ-weight space. All other weight spaces have weights $\mu < \lambda$.*

(3) *Every proper submodule of* $Z(\lambda)$ *has zero* λ*-weight space.*
(4) $Z(\lambda)$ *has a unique maximal submodule (call it* $Y(\lambda)$*), and a unique simple quotient* $V(\lambda)$*.*
(5) $Z(\lambda)$ *is the "universal" cyclic module of highest weight* λ*.*
(6) *If* $v \in Z(\lambda)_\mu$ *is maximal, then* $\mu \leq \lambda$ *in* G*, and* $[Z(\lambda) : V(\mu)] > 0$*.*
(7) *The center* $\mathfrak{Z}(A)$ *acts by the same central character* χ_λ *on all simple objects* $V(\lambda)$*, and this is constant across each block. Namely,* $\mu \in S^3(\lambda) \Rightarrow \chi_\mu = \chi_\lambda$*.*
(8) *On* $\mathfrak{Z}(A)$*, the Harish-Chandra projection* ξ *is an algebra map, and* $\chi_\lambda = \lambda \circ \xi$*.*

We now introduce the following terminology:

Definition 4.6. A is an RTA.
(1) Given $\lambda \in G$, define $S^4(\lambda) := \{\mu \in G : \chi_\mu = \chi_\lambda \text{ on } \mathfrak{Z}(A)\}$.
(2) A satisfies *Condition (S4)* if $S^4(\lambda)$ is finite for each $\lambda \in G$.

Corollary 4.7. $S^3(\lambda) \subset S^4(\lambda)$ *for all* $\lambda \in G$*, so we have the following implications among the Conditions (S):* $(S4) \Rightarrow (S3) \Rightarrow (S2) \Rightarrow (S1)$*.*

We will use this to show that semisimple Lie algebras, quantum groups, and symplectic oscillator algebras are all examples of strict Hopf RTAs, that satisfy all these Conditions (S) - whereas extended (centerless) Heisenberg algebras do not satisfy any of them.

5. Summary of results for RTAs

We now mention our results for HRTAs, that hold in greater generality for RTAs as well. However, since the rest of this paper only deals with HRTAs, we will henceforth assume that A is an HRTA.

Theorem 5.1. *Recall that* $(S4) \Rightarrow (S3) \Rightarrow (S2) \Rightarrow (S1)$ *for an HRTA* A*.*
(1) *If* A *satisfies Condition (S1), then* $\mathcal{O}$ *is finite length, and hence splits into a direct sum of blocks* $\mathcal{O}(\lambda)$*, each of which is abelian and self-dual.*
(2) *If* A *satisfies Condition (S2), then each block has enough projectives, each with a filtration whose subquotients are Verma modules.*
(3) *If* A *satisfies Condition (S3), then each block* $\mathcal{O}(\lambda)$ *is a highest weight category, equivalent to the category* $(\mathrm{Mod}-B)^{fg}$ *of finitely generated right modules over a finite-dimensional* k*-algebra* $B = B_\lambda$*. Many different notions of block decomposition all coincide, including the one that uses the categorical definition of linking.*

Remark 5.2. (For the definition of a highest weight category, see [CPS].) Thus, if A satisfies (S3), then (by the theorem,) each block has enough projectives, finite cohomological dimension, *tilting modules* (i.e., modules simultaneously filtered in $\mathcal{O}$ by standard as well as costandard subquotients - see [Rin, Don]), and the property of *BGG reciprocity*; these properties pass

on to $\mathcal{O}$. Thus, Theorem 5.1 implies that the algebras B_λ are *BGG algebras* (see [Irv]).

We now state a result on the blocks of category $\mathcal{O}$, that was stated and (essentially) explained in [GK].

Theorem 5.3. *Suppose that in every block $\mathcal{O}(\lambda)$, the only finite-dimensional simple $V(\lambda)$'s are in one fiber $\pi^{-1}(\lambda_0)$ (i.e., $\pi(\lambda) = \lambda_0$ for all such λ). Then every finite-dimensional module in $\mathcal{O}$ is completely reducible.*

The last result we state in this general setup, relates the category $\mathcal{O}$ over a tensor product $A = \otimes_{i=1}^n A_i$ of RTAs A_i, with the individual categories $\mathcal{O}_{A_i}$. Once more, if any $(G_i, *_i)$ is k-linear, we will need $\mathrm{char}(k) = 0$.

Theorem 5.4. *Suppose A_i is a HRTA for $1 \leq i \leq n$; then so is $A := \otimes_i A_i$.*

(1) *A module $V \in \mathcal{O}_A$ is simple if and only if $V = \otimes_i V_i$, with V_i simple in $\mathcal{O}_{A_i}$ for all i.*
(2) *The Conditions (S) hold for A if and only if they hold for all A_i. More precisely, for $1 \leq n \leq 4$,*
$$S_A^n((\lambda_1, \dots, \lambda_n)) = \times_i S_{A_i}^n(\lambda_i),$$
as subsets of $G = \times_i G_i$ and $G_0 = \times_i (G_i)_0$ respectively.
(3) *Complete reducibility (for finite-dimensional modules) holds in $\mathcal{O}_A$ if and only if it holds in every $\mathcal{O}_{A_i}$.*

In the paper [Kh2], we prove the above results, or suitable analogues, not for the above setup, but for the more general setup involving $A \rtimes \Gamma$, where Γ is a finite group acting "nicely" on A.

5.1. **Remarks on the Conditions (S).** Each of these conditions (S) yields nontrivial information about the representation theory in $\mathcal{O}$ (see Theorem 5.1). Hence, while discussing several of the examples below, we analyze whether or not they satisfy various Conditions (S). Moreover, even if an algebra A does not satisfy condition (S4), one can ask if $\mathcal{O}$ over it has enough projectives, or finite length. (To be sure, all the Conditions (S) were only sufficient.)

Similarly, the center and the Harish-Chandra projection - or some other properties of A - are well-understood. In this case, verifying condition (S4) may be easy - and gives us information as in [BGG1]. (See the examples of semisimple Lie algebras, "fat" quantum groups, and symplectic oscillator algebras below.)

On the other hand, it may be easy to verify that condition (S1) does not hold - which would imply, by Corollary 4.7, that none of the Conditions (S) hold. See Theorem 6.2; it has been applied to some examples in §7.

Next, *all* the conditions (S) (and not just (S4)) are quite natural to consider, in the context of the category $\mathcal{O}$. For example, $S^1(\lambda)$ is closely related to the set of possible submodules of a Verma module $Z(\lambda)$.

Finally, these conditions are all functorial in nature - as in Theorem 5.4.

6. A special class of examples

We now discuss a lot of examples of RTAs, most of which are actually strict Hopf RTAs that are well-known and widely studied in representation theory. Some are old and familiar examples, others are "modern", and yet others are a combination of the old and the new, such as "fat quantum groups".

We start by defining a special class of strict HRTAs, and then give many examples of this. We then show a result, that helps in showing that Condition (S1) (and hence, all Conditions (S)) fails, in some of those examples.

6.1. **Lie algebras with regular triangular decomposition.** These Lie algebras $\mathfrak{g}$ have been defined in [RCW, MP]; let us recall the definition here. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ be the triangular decomposition, where all objects are nonzero Lie subalgebras of $\mathfrak{g}$, with $\mathfrak{h}$ abelian. We further assume that $(\mathrm{char}(k) = 0$, and that)

1. $\mathfrak{g}_+$ is stable under the adjoint action of $\mathfrak{h}$, and admits a weight space decomposition, with finite-dimensional weight spaces.
2. All weights α (for $\mathfrak{g}_+$ with respect to $\mathrm{ad}\,\mathfrak{h}$) lie in $Q_+ \setminus \{0\}$, where Q_+ denotes a free abelian semigroup with basis $\{\alpha_j\}_{j\in J}$; this basis consists of linearly independent vectors in $\mathfrak{h}^*$.
3. There exists an anti-involution σ of $\mathfrak{g}$ that sends $\mathfrak{g}_+$ to $\mathfrak{g}_-$ and preserves $\mathfrak{h}$ pointwise.

We further assume here, that the index set J is finite. Such a Lie algebra is also denoted henceforth as an *RTLA*.

We claim that we have the *UEA functor* $\mathfrak{U}$, that takes an RTLA $\mathfrak{g}$ to the strict Hopf RTA $\mathfrak{U}\mathfrak{g}$. To see this, take

$$A = \mathfrak{U}\mathfrak{g},\ B_\pm = \mathfrak{U}\mathfrak{g}_\pm,\ H = H_0 = \mathfrak{U}\mathfrak{h} = \mathrm{Sym}\,\mathfrak{h},\ i = \sigma,\ \Delta = \{\alpha_j : j \in J\}.$$

Hence $G = G_0 = \mathrm{Hom}_{k-alg}(\mathrm{Sym}\,\mathfrak{h}, k) = \mathrm{Hom}_k(\mathfrak{h}, k) = \mathfrak{h}^*$ by universality. Note that $H \otimes B_\pm$ are subalgebras, because $\mathfrak{g}_\pm$ are stable under $\mathrm{ad}\,\mathfrak{h}$. Moreover, the property of weights holds, because

$$h(a_\theta m_\lambda) = [h, a_\theta]m_\lambda + a_\theta h m_\lambda = \theta(h)a_\theta m_\lambda + a_\theta(\lambda(h)m_\lambda) = (\theta+\lambda)(h)a_\theta m_\lambda.$$

The other (strict Hopf) RTA axioms are also easy to verify, using the property of weights and the PBW theorem. Moreover, for regularity, it suffices to find a countable set of generators for $\mathfrak{g}_+$, each with weight in $Q_+ \setminus \{0\}$, and only finitely many for each given weight. (This is our approach in the examples below.) We summarize our verification in

Proposition 6.1. *If $\mathfrak{g}$ is an RTLA, then $\mathfrak{U}\mathfrak{g}$ is a strict Hopf RTA. If $\mathfrak{g}_i$ is an RTLA for $1 \le i \le n$, and $\mathfrak{h}'$ is an abelian Lie algebra, then $\mathfrak{h}' \oplus \bigoplus_{i=1}^n \mathfrak{g}_i$ is an RTLA as well.*

6.2. **Special case.** We show a small result here, which is useful in analyzing several examples of RTLAs below.

Theorem 6.2. *Suppose A is a strict RTA, and there exists $\lambda \in G$ such that $[B_+, B_-] \subset A \cdot N_+ \oplus B_- \cdot \ker \lambda$.*

(1) *If $\dim(B_-)_{-\alpha} \neq 0$, then $\dim V(-\alpha * \lambda) = 1$.*
(2) *$Z(\lambda)$ has a Jordan-Holder series with one-dimensional subquotients. Thus, $[Z(\lambda) : V(-\alpha * \lambda)] = \dim(B_-)_{-\alpha}$ for all $\alpha \geq 0$.*
(3) *Condition (S1) (and hence all Conditions (S)) fails if $\dim B_- = \infty$.*
(4) *In this setup, the following are equivalent:*
   (a) *Every finite-dimensional $M \in \mathcal{O}$ is completely reducible.*
   (b) *$\dim B_- = 1$, i.e., $B_- = k$ (or $N_- = 0$).*
   (c) *Condition (S3) holds and $[B_+, B_-] \subset A \cdot N_+$.*
   (d) *$S^3(\lambda) = \{\lambda\}$ for all $\lambda \in G$.*

For example, if $A = \mathfrak{U}\mathfrak{g}$ for an RTLA $\mathfrak{g}$, and $\mathfrak{g}_+, \mathfrak{g}_-$ commute, then $[B_+, B_-] = 0$, so every $\lambda \in G = \mathfrak{h}^*$ works. Moreover, $\dim \mathfrak{U}\mathfrak{g}_- = \infty$ if $\mathfrak{g}_- \neq 0$.

Proof. The key observation is that if v_λ is maximal in the Verma module $Z(\lambda)$, then for all $b_- \in B_-$, the vector $b_- v_\lambda$ is also maximal. This is because for any $n_+ \in N_+$, we compute:

$$n_+(b_- v_\lambda) = n_+ b_- \cdot v_\lambda \in b_- n_+ v_\lambda + A N_+ v_\lambda + A \cdot \ker \lambda \cdot v_\lambda = 0$$

We now show the various parts.

(1) If $0 \neq b_\alpha \in (B_-)_{-\alpha}$, define $w = b_\alpha v_\lambda$. Now note that $b_- w = (b_- b_\alpha) v_\lambda$ is maximal for all $b_- \in B_-$, by the key observation above, whence $N_- w$ is indeed the maximal proper submodule of $B_- w$. Since $Z(-\alpha * \lambda) \twoheadrightarrow B_- \cdot w \twoheadrightarrow B_- w / N_- w \twoheadrightarrow V(-\alpha * \lambda)$, and this quotient is one-dimensional by character theory, hence $V(-\alpha * \lambda) \cong k$.

(2) For any α, let U_α be a (weight) basis of $(B_-)_{-\alpha}$. Now select any maximal chain

$$U_{0,\alpha} = \emptyset \subset U_{1,\alpha} \subset \cdots \subset U_{n_\alpha,\alpha} = U_\alpha$$

of subsets of U_α, where $n_\alpha = |U_\alpha| = \dim(B_-)_{-\alpha}\ \forall \alpha$, and $|U_{j,\alpha}| = j$ for all j, α. For each $\alpha \geq 0$ and $0 \leq j \leq n_\alpha$, we now define the vector subspace $M_{\alpha,j}$ of $Z(\lambda)$, by taking the span of $\{b_- v_\lambda\}$, over all $b_- \in U_{j,\alpha} \cup \bigcup_{\beta \nleq \alpha} U_\beta$.

It is then clear by the key observation and character theory, that every $M_{\alpha,j}$ is a submodule of $Z(\lambda)$, and that $M_{\alpha,j} \subset M_{\beta,k}$ if and only if $\alpha > \beta$, or $\alpha = \beta$ and $j < k$. Moreover, $M_{\alpha,r+1}/M_{\alpha,r} \cong V(-\alpha * \lambda)$ for all α, r.

It is now easy to write down a Jordan-Holder series for $Z(\lambda)$ using only the $M_{\alpha,r}$'s, that is of the desired form.

(3) If $\dim B_- = \infty$, then by the regularity condition (RTA6), the set $W := \{\theta < 0_G : (B_-)_\theta \neq 0\}$ is infinite; now use (RTA6) once more to conclude that $\pi(W)$ is also infinite. Since $[Z(\lambda) : V(\theta * \lambda)] \neq 0$ for each θ now (by a previous part), hence $S^1(\lambda) \supset \{\pi(\theta) * \pi(\lambda) : \theta \in W\}$. By (RTA4), $S^1(\lambda)$ is also infinite now, so Condition (S1) fails, and by Corollary 4.7, so do all Conditions (S).

(4) We prove a series of cyclic implications:
(a) $\Rightarrow$ (b):
We prove the contrapositive. If $\dim B_- \neq 1$, choose a minimal $\alpha > 0$ such that $\dim(B_-)_{-\alpha} \neq 0$. Using the notation above, we observe that $Z(\lambda) \supset M_{\alpha,n_\alpha} \supset M_{\alpha,n_\alpha-1}$, with both subquotients being one-dimensional, by the minimality of α. If we now define $V = Z(\lambda)/M_{\alpha,n_\alpha-1}$, then we get

$$0 \to V(-\alpha * \lambda) \to V \to V(\lambda) \to 0.$$

If this is to split in $\mathcal{O}$, then the complement must contain a weight vector of weight λ. Thus the image of v_λ is in the complement, and since this generates all of V, we get that complete reducibility fails, as claimed.

(b) $\Rightarrow$ (c),(d):
First, $Z(\lambda) = V(\lambda) \cong k$, whence $S^3(\lambda) = \{\lambda\}$ for all λ. Thus Condition (S3) holds, and we of course have $[B_+, B_-] = [B_+, k] = 0 \subset A \cdot N_+$.

(d) $\Rightarrow$ (a): This follows by Theorem 5.3.

(c) $\Rightarrow$ (b):
Let us show the contrapositive. Suppose $[B_+, B_-] \subset A \cdot N_+$ and $\dim B_- > 1$. Choose any $\alpha > 0$ such that $(B_-)_{-\alpha} \neq 0$. Then $[Z(\lambda) : V(-\alpha * \lambda)] > 0$ for all λ, from a part above. Thus $S^3(\lambda) \supset \{-n\alpha * \lambda : n \in \mathbb{Z}_{\geqslant 0}\}$, and Condition (S3) fails.

□

7. Examples of RTLAs

Here are some examples of Lie algebras with regular triangular decomposition.

Example 7.1 (*Symmetrizable Kac-Moody (and semisimple) Lie algebras)***.** See [HK, §2.1] for the definition and basic properties of $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Now $G = \mathfrak{h}^*$ as above, and if $\mathfrak{g}$ is complex semisimple, Harish-Chandra's theorem implies that $S^4(\lambda) = W \bullet \lambda \; \forall \lambda \in \mathfrak{h}^*$ (the twisted Weyl group orbit). Thus, all Conditions (S) hold (by Corollary 4.7).

We now mention two generalizations of these, which are also RTLAs.

Example 7.2 (*Contragredient Lie algebras)***.** These are a family of Lie algebras defined in [KK], that can be verified to be RTLAs. Kac and Kazhdan

also proved the *the Shapovalov determinant formula* for them. We note that we encounter a similar definition of "fat quantum groups" below.

Example 7.3 (*Some (symmetrizable) Borcherds algebras and central extensions)***.** These are defined in [Bo1, Bo2]; we rewrite the definitions here, with some additional assumptions. We assume that A is a *symmetrizable Borcherds-Kac-Moody (BKM) matrix* (see [Wa, §2.1]), say of finite size n. Thus, $A \in \mathfrak{gl}_n(\mathbb{R})$ is symmetrizable (there exists a diagonal matrix D with positive eigenvalues, such that DA is symmetric); we have $a_{ii} \leq 0$ or $a_{ii} = 2$ for all i; $a_{ij} \leq 0$ for all $i \neq j$; and $a_{ij} \in \mathbb{Z}$ whenever $a_{ii} = 2$.

(Note that here A is taken to be a matrix and not the algebra $\mathfrak{Ug}$; we do not mention $\mathfrak{Ug}$ in this example, just as we did not, in the previous one.)

We now define the *universal Borcherds algebra* (see [Ga]) $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}(A)$ to be generated by e_i, f_i, h_{ij} for $1 \leq i, j \leq n$, satisfying:

(1) $[e_i, f_j] = h_{ij}, \ [h_{ij}, e_k] = \delta_{ij} a_{ik} e_k, \ [h_{ij}, f_k] = -\delta_{ij} a_{ik} f_k$ for all i, j, k;
(2) $(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = (\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0$ whenever $a_{ii} = 2$ and $i \neq j$;
(3) $[e_i, e_j] = [f_i, f_j] = 0$ whenever $a_{ij} = 0$.

Also define the *Borcherds algebra* $\mathfrak{g} = \mathfrak{g}(A) := \widehat{\mathfrak{g}}(A)$ modulo: $h_{ij} = 0 \ \forall i \neq j$. Then $\widehat{\mathfrak{g}}(A)$ is a central extension of $\mathfrak{g}(A)$, and they are both RTLAs, under the following **additional assumptions**:

(B1) no column of A is zero;
(B2) the $\mathbb{Z}_{\geqslant 0}$-span of the columns of A (i.e., the semigroup Q_+), is freely generated by a subset Δ of columns of A; and
(B3) $h_{ij} = h_{ji}$ for all i, j.

Example 7.4 (*The Virasoro and Witt algebras)***.** See [FF], for example.

Example 7.5 (*(Centerless) Heisenberg algebras extended by derivations)***.** Both these and the (centerless) Virasoro algebras can be found in [MP], for instance. One checks that all Conditions (S) fail by Theorem 6.2, if $V \neq 0$.

Example 7.6 (*Certain quotients of preprojective algebras of loop-free quivers)***.** Let Q be a finite acyclic quiver (i.e., containing no loops or oriented cycles) with path algebra $kQ = \oplus_{n \geq 0} (kQ)_n$, where each summand has a basis consisting of (oriented) paths in Q of length n. Thus $(kQ)_0$ and $(kQ)_1$ have bases I of *vertices* e_i and E of *edges* a respectively. Assume $I, E \neq \emptyset$. Now construct the *double* $\overline{Q}$ of Q, by adding an "opposite" edge a^* for each $a \in E$.

The subquover Q^* is defined with vertices I and edges a^*. Now define $\mathfrak{g} = k\overline{Q}/(a'a^*, a^*a' : a' \in (kQ)_1, a^* \in (kQ^*)_1)$. This is an associative algebra, and a quotient of the *preprojective algebra* introduced in [GP], namely, $k\overline{Q}/(\sum_{a \in E} [a, a^*])$. One uses the associative algebra structure to show that $\mathfrak{g}$ is an RTLA, using: $\mathfrak{g}_+ = \oplus_{n>0} (kQ)_n, \ \mathfrak{h} = (kQ)_0$, and $\mathfrak{g}_- = \oplus_{n>0} (kQ^*)_n$. Moreover, $[\mathfrak{g}_+, \mathfrak{g}_-] = 0$, whence all Conditions (S) fail by Theorem 6.2.

Remark 7.7 (*Toroidal Lie algebras)***.** These are defined (see [BM, §0]) to be the universal central extension of $R_n \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a simply laced Lie algebra and $R_n = k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. The central extension is by $\mathfrak{Z} = \Omega^1 R_n / dR_n$.

Clearly, the regularity condition fails here, so that toroidal Lie algebras are not RTLAs. We can, however, look at a related algebra, namely $\mathfrak{U}\mathfrak{g} \otimes R_n$. By the above result, this is an HRTA. If the central extension above splits, then $\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{Z}) \otimes R_n$ is also an HRTA.

8. Fat quantum groups for Kac-Moody Lie algebras

Our next class of examples also consists of strict Hopf RTAs. Suppose A is a Cartan matrix, corresponding to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ (say over $\mathbb{C}$). Let us also fix a ground field k, and a nonzero $q \in k$ that is not a root of unity. That A is symmetrizable means that there exist positive integers $\{s_i : i \in I\}$ such that $s_i a_{ij} = s_j a_{ji}$.

The *goal* in this section is to

- construct a family of "fat quantum groups" associated to A, and
- prove that these quantum groups satisfy Condition (S4), and hence, all the Conditions (S) - whence $\mathcal{O}$ over them, has a block decomposition into highest weight categories.

For instance, these would include quantum groups that use neither the co-root lattice nor the co-weight lattice, but some intermediate lattice.

8.1. The construction and triangular decomposition.

To define a quantum group over $\mathfrak{g}(A)$, we need (at least)

- an abelian group Γ, that will play the role of the "middle part" U_q^0. We will freely identify Γ with q^Γ (and q^{0_Γ} with 1); this creates no problems, since q is not a root of unity.
- for a finite set I (of "simple roots"), distinguished elements $K_i \in \Gamma$, that generate a free abelian group of rank $|I|$, denoted by $Q^\vee$.
- a *group of weights* $G = \mathrm{Hom}_{group}(\Gamma, k^\times)$ that contains distinguished characters $\{\nu_j : j \in I\}$, which satisfy: $\nu_j(K_i) = q^{s_i a_{ij}}\ \forall i, j \in I$.

For example, if we follow [HK, §3.1] or [Jo, §3.2.10], then $\Gamma = P^\vee$, the co-weight lattice inside $\mathfrak{h}$, and $\nu_i(q^h) = q^{\alpha_i(h)}$, for the simple roots $\alpha_i \in \mathfrak{h}^*$; moreover, $K_i = q^{s_i h_i}$, where $h_i = [e_i, f_i]$ in $\mathfrak{U}\mathfrak{g}$.

On the other hand, we may choose $\Gamma = Q^\vee$, the co-root lattice inside $\mathfrak{h}$, and $\nu_i = q^{\alpha_i}$; this is the approach followed in [Ja, §4.2] (when A is of finite type). In what follows, we will freely identify α_i with q^{α_i}, since we only deal with quantum groups here, and q is not a root of unity.

Thus, we now introduce the main construction in this analysis. In some sense, it "quantizes" the contragredient Lie algebras defined above (or in [KK]), after removing some of the assumptions therein.

Definition 8.1. Suppose we are given an abelian group $\Gamma \supset \mathbb{Z}^I \cong Q^\vee = \langle K_i : i \in I \rangle$ and characters $\{\nu_j : j \in I\}$ such that $\nu_j(K_i) = q^{s_i a_{ij}}$.

(1) Define the *fat quantum group* $\mathfrak{U}_{q,A}(\Gamma,\nu)$ to be the k-algebra generated by $q^\Gamma = \{q^g : g \in \Gamma\}$ and $\{e_i, f_i : i \in I\}$, modulo the relations:

$$q^g \cdot q^{g'} = q^{g+g'}, \qquad q^g e_i q^{-g} = \nu_i(g) e_i, \qquad q^g f_i q^{-g} = \nu_i(g^{-1}) e_i,$$

$$q^{0_\Gamma} = 1, \qquad [e_i, f_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q^{s_i} - q^{-s_i}},$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \binom{1-a_{ij}}{l}_{q^{s_i}} e_i^{1-a_{ij}-l} e_j e_i^l = 0, \qquad \text{(q-Serre-1)}$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \binom{1-a_{ij}}{l}_{q^{s_i}} f_i^{1-a_{ij}-l} f_j f_i^l = 0. \qquad \text{(q-Serre-2)}$$

(2) Also define $U_q(\mathfrak{g})$ to be $\mathfrak{U}_{q,A}(Q^\vee, \{q^{\alpha_i} : i \in I\})$.
(3) Define $U_q^\pm(\Gamma,\nu)$ to be the subalgebras of $\mathfrak{U}_{q,A}(\Gamma,\nu)$ generated by the e_i's and f_i's respectively.
(4) Given Γ, ν, and a subgroup $\Gamma \supset \Gamma' \supset Q^\vee$, define $\mathrm{Res}^\Gamma_{\Gamma'}(A)$ to be the subalgebra of $\mathfrak{U}_{q,A}(\Gamma,\nu)$ generated by e_i, f_i, and Γ'.
(5) One has the *root lattice*[1] $Q = \sum_i \mathbb{Z}\nu_i \subset G = \mathrm{Hom}_{group}(\Gamma, k^\times)$, and the usual adjoint action of Γ on $\mathfrak{U}_{q,A}(\Gamma,\nu)$ gives a direct sum decomposition into weight spaces (with all weights in Q).
(6) Also define $\mathrm{Ind}^\Gamma_{\Gamma'}(A)$ to be the algebra $\mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'}) \otimes_k k^S$, where $S \cong \Gamma/\Gamma'$ is the lift to Γ, of a set of coset representatives of Γ' in Γ (including 1, say). Let us denote the basis vector $[s]$ in k^S as q^s. Then the multiplication in $\mathrm{Ind}^\Gamma_{\Gamma'}(A)$ is given on a spanning set by

$$(u_\lambda \otimes q^s) \cdot (u_{\lambda'} \otimes q^{s'}) = \lambda'(s) u_\lambda u_{\lambda'} \otimes q^{s+s'},$$

where $s, s' \in S$, and given a weight λ of Γ, u_λ is in the λ-weight space for the adjoint action of Γ.

Remark 8.2. Note that these quantum groups can be defined for Γ any intermediate lattice between $Q^\vee$ and $P^\vee$. Moreover, Γ may even have torsion elements, in which case it does not embed into $\mathbb{Q} \otimes_{\mathbb{Z}} Q^\vee \subset \mathfrak{h}$. Thus, there may not be a way to define a bilinear form (and hence, a Hopf pairing) on it, as usual.

We now have the first results (the "Structure Theorems") on any such quantum group.

Theorem 8.3. Γ, ν *as above (and k, q, A fixed).*

(1) *We have the surjection:* $U_q^-(\Gamma,\nu) \otimes k\Gamma \otimes U_q^+(\Gamma,\nu) \twoheadrightarrow \mathfrak{U}_{q,A}(\Gamma,\nu)$.

[1] We abuse notation even if Q is not a lattice in G.

(2) $\mathfrak{U}_{q,A}(\Gamma, \nu)$ *has a Hopf algebra structure, with the comultiplication* Δ*, counit* ϵ*, and antipode* S *given on generators by*

$$\begin{aligned}
\Delta(q^g) &= q^g \otimes q^g, \\
\Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i, \qquad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i, \\
\epsilon(q^g) &= 1, \qquad \epsilon(e_i) = \epsilon(f_i) = 0, \\
S(q^g) &= q^{-g}, \qquad S(e_i) = -e_i K_i, \qquad S(f_i) = -K_i^{-1} f_i.
\end{aligned}$$

(3) $\mathfrak{U}_{q,A}(\Gamma, \nu)$ *has an involution* T *that satisfies:*

$$T: \qquad q^g \leftrightarrow q^{-g}, \qquad e_i \leftrightarrow f_i,$$

and restricted to $U_q^+(\Gamma, \nu)$ *gives an algebra isomorphism* : $U_q^+(\Gamma, \nu) \to U_q^-(\Gamma, \nu)$.

(4) $TST = S^{-1} \neq S$*, whence* $ST \neq TS$ *are anti-involutions i on* $\mathfrak{U}_{q,A}(\Gamma, \nu)$*, that restrict to the identity on* H.

Proof. The first part is clear; the second and third parts can be shown by imitating the proofs and arguments found in [HK, §3.1]. Since both TST and S^{-1} are k-algebra anti-automorphisms, the last part follows by checking that they agree on generators. $\square$

Now suppose that the ν_i's are $\mathbb{Z}$-linearly independent characters (for instance, this always holds when A is nonsingular, since the ν_i's restrict to linearly independent characters on $Q^\vee$). Then as is standard, we introduce a *partial order* on $G = \mathrm{Hom}_{group}(\Gamma, k^\times)$: $\chi \geq \chi'$ if and only if $\chi - \chi' \in Q_+ = \mathbb{Z}_{\geqslant 0}\Delta$, where Δ is the set of *simple roots* $\Delta = \{\nu_i : i \in I\}$.

Theorem 8.4. $\mathfrak{U}_{q,A}(\Gamma, \nu)$ *as above. Assume that the* ν_i*'s are linearly independent in* G.

(1) *Then* $\mathfrak{U}_{q,A}(\Gamma, \nu)$ *has the* triangular decomposition

$$\mathfrak{U}_{q,A}(\Gamma, \nu) \cong U_q^-(\Gamma, \nu) \otimes U_q^0 \otimes U_q^+(\Gamma, \nu) \cong B_- \otimes H \otimes B_+,$$

where $B_\pm := U_q^\pm(\Gamma, \nu)$*, and* $H = U_q^0 = k\Gamma$*, the group algebra.*

(2) *Moreover,* $\mathfrak{U}_{q,A}(\Gamma, \nu)$ *is a strict Hopf RTA.*

(3) *If* $Q^\vee \subset \Gamma' \subset \Gamma$ *is a chain of subgroups, then* $\mathrm{Res}^\Gamma_{\Gamma'}(A)$ *has a triangular decomposition. If the* $\nu_i|_{\Gamma'}$ *are also linearly independent, then so does* $\mathrm{Ind}^\Gamma_{\Gamma'}(A)$*, since we have*

$$\mathrm{Ind}^\Gamma_{\Gamma'}(A) \cong \mathfrak{U}_{q,A}(\Gamma, \nu) \supset \mathrm{Res}^\Gamma_{\Gamma'}(A) \cong \mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'}).$$

(4) *If* $Q^\vee \subset \Gamma' \subset \Gamma$ *and the* ν_i*'s are linearly independent on* Γ'*, then the centers of the two quantum groups have the same property:* $\mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'})) = \mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma, \nu)) \cap \mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'})$.

For example, if we take $\Gamma = P^\vee$ when A is symmetrizable, or $Q^\vee \subset \Gamma \subset P^\vee$ (i.e., an "intermediate lattice") when A is nonsingular (e.g., of finite type), then these theorems say that $U_q(\mathfrak{g}) := \mathfrak{U}_{q,A}(Q^\vee, \{\alpha_i\})$ is a Hopf algebra with

triangular decomposition. Moreover, if A is of finite type (in particular, nonsingular), then $U_q(\mathfrak{g})$ has nontrivial center, hence so does $\mathfrak{U}_{q,A}(\Gamma, \nu)$.

Proof.

(1) This follows by imitating the "remaining" (from the preceding proof) arguments found in [HK, §3.1].

(2) The previous part and the last part of Theorem 8.3 already show some of the axioms - for example, the property of weights:

$$q^h a_\theta m_\lambda = q^h a_\theta q^{-h} \cdot q^h m_\lambda = q^{\theta(h)} a_\theta \cdot q^{\lambda(h)} m_\lambda = q^{(\theta+\lambda)(h)} a_\theta m_\lambda.$$

The rest is standard for quantum groups (see, e.g., [Ja]): one shows that $B_\pm = \bigoplus_{\lambda \in \pm \mathbb{Z}_{\geqslant 0}\Delta} (B_\pm)_\lambda$. It is easy to check that each summand here is a finite-dimensional weight space for the adjoint action of $H = k\Gamma$, and $(B_\pm)_0 = k$.

(3) The first part is easy: given any subgroup $Q^\vee \subset \Gamma' \subset \Gamma$, note that $U_q^-(\Gamma, \nu) \otimes \Gamma' \otimes U_q^+(\Gamma, \nu)$ is a subalgebra of $\mathfrak{U}_{q,A}(\Gamma, \nu)$, with triangular decomposition (by the first part of this result). Since this subalgebra is generated by e_i, f_i, and Γ', it equals $\mathrm{Res}^\Gamma_{\Gamma'}(A)$, and we are done.

Now assume that the $\nu_i|_{\Gamma'}$'s are ("multiplicatively") $\mathbb{Z}$-linearly independent. Then $\mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'})$ has a triangular decomposition from above, whence so does $\mathrm{Ind}^\Gamma_{\Gamma'}(A)$. Moreover, the ν_i's are also $\mathbb{Z}$-linearly independent over Γ, whence both $\mathfrak{U}_{q,A}(\Gamma, \nu)$ and $\mathrm{Res}^\Gamma_{\Gamma'}(A)$ have triangular decompositions.

Moreover, $\mathrm{Res}^\Gamma_{\Gamma'}(A)$ is generated by e_i, f_i, and Γ', and satisfies the defining relations for $\mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'})$; this gives us a surjection from one algebra to the other. The story is similar for $\mathfrak{U}_{q,A}(\Gamma, \nu)$ and $\mathrm{Ind}^\Gamma_{\Gamma'}(A)$, but the surjection is "reversed" here. We thus have the following "commuting" diagram of algebras, in terms of their triangular decompositions:

$$\begin{tikzcd}
U_q^-(\Gamma, \nu) \otimes k\Gamma' \otimes U_q^+(\Gamma, \nu) \arrow[d, "\hookrightarrow"'] & U_q^-(\Gamma', \nu) \otimes k\Gamma' \otimes U_q^+(\Gamma', \nu) \arrow[l, "\twoheadleftarrow"'] \arrow[d, "\hookrightarrow"'] \\
U_q^-(\Gamma, \nu) \otimes k\Gamma \otimes U_q^+(\Gamma, \nu) \arrow[r, "\twoheadrightarrow"] & U_q^-(\Gamma', \nu) \otimes k\Gamma \otimes U_q^+(\Gamma', \nu)
\end{tikzcd}$$

Note that all maps in the above square respect the triangular decomposition. Thus, the above commuting square induces the following commuting square(s):

$$\begin{tikzcd}
U_q^\pm(\Gamma, \nu) \arrow[d, "\cong"'] & U_q^\pm(\Gamma', \nu|_{\Gamma'}) \arrow[l, "\twoheadleftarrow"'] \arrow[d, "\cong"'] \\
U_q^\pm(\Gamma, \nu) \arrow[r, "\twoheadrightarrow"] & U_q^\pm(\Gamma', \nu|_{\Gamma'})
\end{tikzcd}$$

But since the ν_i's are $\mathbb{Z}$-linearly independent on Γ', Γ (whence $\mathfrak{U}_{q,A}(\Gamma, \nu), \mathfrak{U}_{q,A}(\Gamma', \nu)$ are strict Hopf RTAs), hence $U_q^\pm(\Gamma, \nu)$ and

$U_q^{\pm}(\Gamma', \nu)$ are modules with finite-dimensional weight spaces for the adjoint action of Γ (or Γ'). Thus, we can talk of their formal characters; equating these indicates (given the linear independence of the ν_i's) that all maps in the last square are isomorphisms.

(4) It suffices to show that $\mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'})) \subset \mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma, \nu))$. Since z commutes with Γ', it has weight 0 in $\mathfrak{U}_{q,A}(\Gamma', \nu|_{\Gamma'})$, and hence also in $\mathfrak{U}_{q,A}(\Gamma, \nu)$ (since the weight space decompositions of $\mathfrak{U}_{q,A}(\Gamma', \nu) \hookrightarrow \mathfrak{U}_{q,A}(\Gamma, \nu)$ agree). Thus, z commutes with Γ, and since it commutes with each e_i and f_i, z is central in $\mathfrak{U}_{q,A}(\Gamma, \nu)$ too.

$\square$

Remark 8.5.

(1) Here is a standard consequence of the triangular decomposition. For each weight $\mu \in G$, we can now define the *Verma module* $Z(\mu)$ and its unique simple quotient $V(\mu)$ over $\mathfrak{U}_{q,A}(\Gamma, \nu)$. Hence the center of $\mathfrak{U}_{q,A}(\Gamma, \nu)$ acts by characters on all of these, via: $\chi_\lambda = \lambda \circ \xi$, where ξ is the Harish-Chandra homomorphism : $\mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma, \nu)) \to k\Gamma$.

(2) Suppose the Cartan matrix A is of finite type, and we restrict the map ξ to $\mathfrak{U}_{q,A}(Q^\vee, \{\nu_i = q^{\alpha_i}\}) \subset \mathfrak{U}_{q,A}(\Gamma, \nu)$. This restriction is related, but not equal, to the *Harish-Chandra homomorphism* Θ : $\mathfrak{Z}(\mathfrak{U}_{q,A}(Q^\vee, \{q^{\alpha_i}\})) \to k\Gamma$ (in the sense of [Ja, §6.4]); this latter is given by $\Theta := \gamma_{-\rho} \circ \xi$, where ρ is the half-sum of positive roots, and $\gamma_\lambda(K_\mu) := q^{(\lambda,\mu)} K_\mu \ \forall \lambda, \mu$. (We identify $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$ via the Killing form.)

8.2. **Classical limit, and representations.** We merely mention that the classical limit and representation theory (at least, for integrable modules) should be similar to the "classical" case; thus we would expect that the analysis in [HK, Chapter 3] should go through here as well.

Also note, by [HK, Theorem 3.5.4], that complete reducibility holds for integrable modules in $\mathcal{O}$, over $U_q(\mathfrak{g})$. (Thus, we also expect it to hold for any $\mathfrak{U}_{q,A}(\Gamma, \nu)$ with a triangular decomposition.)

8.3. **Central characters.** We are now ready to analyze central characters $\{\chi_\lambda : \lambda \in G\}$. For the results that we prove, we need further restrictions.

Standing assumption. For the rest of this section:

(1) Fix a ground field $\Bbbk$, of characteristic not 2 or 3 (this is for technical purposes, e.g., see [Ja, §4.2]).
(2) Fix $q \in \Bbbk^\times$ that is not a root of unity.
(3) We also need that $\Bbbk^\times$ is a *divisible* abelian group, e.g., when $\Bbbk$ is algebraically closed.
(4) A is a symmetrizable Cartan matrix that is nonsingular (e.g., A is of finite type), and $Q^\vee \subset \mathfrak{h} \subset \mathfrak{g} = \mathfrak{g}(A)$ its co-root lattice. Now ignore $\mathfrak{h}$, and identify $Q^\vee$ with $q^{Q^\vee}$, whence $s_i h_i \mapsto K_i$.

We also denote the group algebra of a group Γ by $\Bbbk[\Gamma]$ here.

Now choose and fix *any* abelian group $\Gamma \supset Q^\vee$, and consider the short exact sequence

$$0 \to Q^\vee \xrightarrow{\iota} \Gamma \to \Gamma/Q^\vee \to 0 \tag{8.6}$$

in the category of abelian groups. Since $\Bbbk^\times$ is divisible - i.e., injective - this gives us

$$0 \to \operatorname{Hom}(\Gamma/Q^\vee, \Bbbk^\times) \to \operatorname{Hom}(\Gamma, \Bbbk^\times) \xrightarrow{\iota^*} \operatorname{Hom}(Q^\vee, \Bbbk^\times) \to 0. \tag{8.7}$$

We now think of the simple roots α_i as elements of $\operatorname{Hom}(Q^\vee, \Bbbk^\times)$, via:

$$\alpha_i(q^h) := q^{\alpha_i(h)}.$$

For convenience, we define $\Gamma\hat{} := \operatorname{Hom}(\Gamma, \Bbbk^\times)$, for any abelian group Γ. Note that the subgroup generated by the α_i's is free because q is not a root of unity in $\Bbbk$. Now lift q^{α_i}, via the injectivity of $\Bbbk^\times$, to any $\nu_i \in (\iota^*)^{-1}(q^{\alpha_i}) \in \Gamma\hat{}$.

Lemma 8.8. *The map $\pi : \alpha_i \mapsto \nu_i$ extends to an isomorphism of free abelian (semi)groups spanned by the respective sets (the latter is in $\Gamma\hat{}$).*

Proof. Evaluating even on $Q^\vee \leftrightarrow q^{Q^\vee}$, one verifies that the ν_i's are $\mathbb{Z}$-linearly independent on Γ. This is because they restrict to α_i's on $Q^\vee$, and the Cartan matrix A is nonsingular by assumption. $\square$

We now come to the "central" result in this part. Define χ_μ (as above) to be the central character by which the center $\mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma, \nu))$ acts on the Verma module $Z(\mu)$. Then the following two results are known:

Theorem 8.9. $\Bbbk, A, \mathfrak{g} = \mathfrak{g}(A)$ *as above. Assume that A is of finite type.*

1. *(*[Jo, Lemma 8.3.2]*.) If $\Gamma = P^\vee$ is the weight lattice, then for $\mathfrak{U}_{q,A}(P^\vee, \{q^{\alpha_i}\})$, two central characters χ_λ, χ_μ are equal, if and only if they are "twisted-conjugate" under the "extended Weyl group":*

$$\lambda \in (W \ltimes (\mathbb{Z}/2\mathbb{Z})^I) \bullet \mu.$$

In other words, λq^ρ and μq^ρ are conjugate under $W \ltimes (\mathbb{Z}/2\mathbb{Z})^I$.

2. *(*[Ja, §6.25-6.26]*.) If $\Gamma = Q^\vee$ and $\nu_i = \alpha_i$ are the "simple roots", then the Harish-Chandra map is an isomorphism*

$$\Theta = \gamma_{-\rho} \circ \xi : \mathfrak{Z}(U_q(\mathfrak{g})) \xrightarrow{\sim} \Bbbk[Q^\vee \cap 2P^\vee]^W.$$

We would like to show a similar condition (to the first part) for general $\mathfrak{U}_{q,A}(\Gamma, \nu)$; we need some finiteness condition, the one we choose being that $[\Gamma : Q^\vee] < \infty$ (but Γ can still contain torsion elements). We also need one small fact, which we quote as a special case of [Muk, Theorem 5.3].

Theorem 8.10 (Nagata, Mumford)**.** *Suppose a finite group W acts on an affine variety X (i.e., its coordinate ring R). Then the map $\Phi : X = \operatorname{Spec}(R) \to X//W := \operatorname{Spec}(R^W)$ (induced by the inclusion $R^W \hookrightarrow R$) is a surjection, that factors through a bijection $\Phi : X/W \to X//W$, where X/W denotes the W-orbits in X.*

Finally, here is our main result.

Theorem 8.11. *Suppose the Cartan matrix A is of finite type, and $[\Gamma : Q^\vee] < \infty$. Choose and fix extensions ν_i of q^{α_i} from $Q^\vee$ to Γ. Then for all $\mu \in \widehat{\Gamma}$, there are at most finitely many $\lambda \in \widehat{\Gamma}$ such that $\chi_\lambda = \chi_\mu$ on $\mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma, \nu))$.*

In particular, Condition (S4), and hence all other Conditions (S) too, hold for $\mathfrak{U}_{q,A}(\Gamma, \nu)$ (by Corollary 4.7).

Proof. Suppose $\chi_\lambda = \chi_\mu : \mathfrak{Z}(\mathfrak{U}_{q,A}(\Gamma, \nu)) \to \mathbb{k}$. Then they agree when restricted (by Theorem 8.4) to $Z := \mathfrak{Z}(U_q(\mathfrak{g}))$. Thus $\lambda \circ \xi = \mu \circ \xi$ on Z, and using Theorem 8.9, $\lambda \circ \gamma_\rho = \mu \circ \gamma_\rho$ on $\mathbb{k}[Q^\vee \cap 2P^\vee]^W \hookrightarrow \mathbb{k}[Q^\vee \cap 2P^\vee] \hookrightarrow \mathbb{k}[Q^\vee] \hookrightarrow \mathbb{k}[\Gamma]$.

Now, $\lambda \circ \gamma_\rho = \mu \circ \gamma_\rho$ on $\widehat{((Q^\vee \cap 2P^\vee)^W)}$, so by the injectivity of $\mathbb{k}^\times$, $\lambda \circ \gamma_\rho$ can be extended to all of $Q^\vee \cap 2P^\vee$. (Note that $Q^\vee \cap 2P^\vee$ is a lattice, so $\operatorname{Spec} k[Q^\vee \cap 2P^\vee] = (k^\times)^{rk(Q^\vee \cap 2P^\vee)}$.) By the Nagata-Mumford Theorem, the set of possible extensions $\{\nu\}$ is a W-orbit, hence finite (thus, $\{\nu \circ \gamma_{-\rho}\}$ is also finite). Thus, $\{\lambda \in \widehat{(Q^\vee \cap 2P^\vee)} : \lambda \circ \gamma_\rho = \mu \circ \gamma_\rho \text{ on } (Q^\vee \cap 2P^\vee)^W\}$ is a finite set.

Now consider the map : $\widehat{\Gamma} \to \widehat{(Q^\vee \cap 2P^\vee)}$. By the injectivity of $\mathbb{k}^\times$, and an analogue of equation (8.7) in this situation, we only need to show that $\Gamma/(Q^\vee \cap 2P^\vee)$ is finite (for then it is a surjection with finite fibers). But this is so, since

$$[\Gamma : Q^\vee \cap 2P^\vee] = [\Gamma : Q^\vee] \cdot [Q^\vee : Q^\vee \cap 2P^\vee] \leq [\Gamma : Q^\vee] \cdot [P^\vee : 2P^\vee] < \infty.$$

To conclude, $\{\lambda \in \widehat{\Gamma} : \chi_\lambda = \chi_\mu\} \subset \{\lambda \in \widehat{\Gamma} : \lambda \circ \gamma_\rho = \mu \circ \gamma_\rho \text{ on } (Q^\vee \cap 2P^\vee)^W\}$, and the latter is a finite set. $\square$

9. Other examples of strict Hopf RTAs

We give two more examples of strict HRTAs here, that are not of the form $\mathfrak{U}\mathfrak{g}$ for $\mathfrak{g}$ an RTLA, except perhaps in a degenerate case.

Example 9.1 (*Symplectic oscillator algebras and their quantized analogues*)**.** These algebras were introduced in [Kh1, GK], and extensively studied in [KT]. (Note that these algebras are merely infinitesimal Hecke algebras corresponding to $(\mathfrak{g}, V) = (\mathfrak{sl}_2, k^2)$.) It was also shown in these works, that they satisfy all the strict Hopf RTA axioms stated above (including Condition (S3) if the deformation is nontrivial).

We briefly discuss the classical version here (from [KT]): H_z is the quotient of $\mathfrak{U}(\mathfrak{sl}_2) \ltimes T(k^2)$ (where $\mathfrak{sl}_2$ acts naturally on $k^2 = kx \oplus ky$), modulo the relations $[x, y] = z$, where $z \in \mathfrak{U}(\mathfrak{sl}_2)$ is a polynomial in the quadratic Casimir element. Moreover, complete reducibility (for finite-dimensional modules) holds in $\mathcal{O}$, if and only if each block contains at most one simple finite-dimensional module. (Thus, the converse to Theorem 5.3 also holds here.) In [KT], it is also shown that similar to complex semisimple Lie algebras,

- Condition (S4) holds for H_z if $z \neq 0$.
- Every central character is of the form χ_λ for some $\lambda \in G$, if k is algebraically closed of characteristic zero (see [H1, Exercise (23.9)]).
- If $z \neq 0$, there are at most finitely many pairwise non-isomorphic simple finite-dimensional objects in $\mathcal{O}$.

Regarding the quantum analogue - note that if the deformation (not quantization) parameter is nontrivial, then the quantum analogue of H_z has trivial center (see [GK, Theorem 11.1]). In particular, it does not satisfy (S4). However, (S3) does hold (see [GK]), so that for both the "classical" and "quantum" nontrivially deformed algebras, $\mathcal{O}$ is a direct sum of highest weight categories.

Example 9.2 (*Regular functions on affine algebraic groups)***.** It is well-known that the category of commutative Hopf algebras is dual to the category of affine algebraic groups. Thus if $\mathbf{G}$ is any affine algebraic group, then $H = \mathbb{C}[\mathbf{G}]$ is a commutative Hopf algebra, and hence $A = H$ is a strict HRTA as well (properties of RTAs of the form $k \otimes H \otimes k$ are explored in Theorem 6.2). Note that H need not be cocommutative in general (since $\mathbf{G}$ need not be commutative).

In general, $\mathcal{O}$ over a commutative k-Hopf algebra $H = k \otimes H \otimes k = \mathfrak{Z}(H)$ satisfies (S4) (and hence (S1)–(S3)), and also is a semisimple category.

10. Infinitesimal Hecke algebras

Finally, we come to one of the main motivations for this paper - an example of Hopf RTAs that are not strict. Infinitesimal Hecke algebras are deformations of $\mathfrak{U}(\mathfrak{g} \ltimes V)$, where $\mathfrak{g}$ is a reductive Lie algebra, and V a finite-dimensional representation of $\mathfrak{g}$.

The first example of infinitesimal Hecke algebras is over $\mathfrak{sl}_2$; these were the symplectic oscillator algebras described in [Kh1], and they are strict Hopf RTAs. The next two classes of examples come from [EGG].

10.1. **Partial examples.** We first mention a general family of infinitesimal Hecke algebras, that are not fully verified to be Hopf RTAs. The main result of this subsection is to show the equivalence between three conditions, one of which relates to Ginzburg's *Generalized Duflo Theorem* [Gi, Theorem 2.3].

In this subsection, $\mathfrak{g}$ is a reductive Lie algebra, with decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{h}_0$ into a (split) semisimple ideal $\mathfrak{g}_0$ and the center $\mathfrak{h}_0$. Also write $\mathfrak{g}_0 = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, the triangular decomposition. Define $\widetilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathfrak{h}_0$, the Cartan subalgebra of $\mathfrak{g}$.

Proposition 10.1. *Let V be a finite-dimensional $\widetilde{\mathfrak{h}}$-semisimple $\mathfrak{g}$-module, so that V is a direct sum of weight spaces $V_{\lambda,\mu}$ for $(\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}_0^*$. Assume that $\{\mu \in \mathfrak{h}_0^* : (0, \mu) \in \Pi(V)\}$ is linearly independent. Now given $\beta_0 \in k$, define the algebra H_{β_0} to be the quotient of $(\mathfrak{U}\mathfrak{g}) \ltimes T(V \oplus V^*)$ by the relations:*

$$[v_1, v_2] = [v_1^*, v_2^*] = 0, \ [v_1, v_1^*] = \beta_0 v_1^*(v_1), \qquad \forall v_1, v_2 \in V, \ v_1^*, v_2^* \in V^*.$$

(1) *Then H_{β_0} satisfies (HRTA1) and (HRTA2).*
(2) *Suppose the set of $(0,\mu) \in \mathfrak{h}^* \times \mathfrak{h}_0^*$ above is a singleton $(0,\lambda_0) \neq 0$ (so $\mathfrak{h}_0 \neq 0$). Then $\sum_i v_i v_i^* + \beta_0 \tau$ is central in H_{β_0} (where $\{v_i\}, \{v_i^*\}$ are dual bases of V, V^* respectively, and $\tau \in \mathfrak{h}_0$ satisfies: $\lambda_0(\tau) = 1$).*

Note that these results can be generalized to the case of $\widetilde{V}$, an $\widetilde{\mathfrak{h}}$-semisimple $\mathfrak{g}$-module with a $\mathfrak{g}$-invariant symplectic form. We obtain very general results - in fact, a characterization - below, that address (HRTA2).

Proof.

(1) We use some results from [Kh3] to prove the axiom (HRTA1). We note that H_{β_0} is an example of a *Drinfeld-Hecke algebra* generated by $V \oplus V^*$ over $\mathfrak{Ug}$. Thus, H_{β_0} is a flat deformation (i.e., has a "PBW-type property") if and only if the bracket map $[-,-] : \wedge^2(V \oplus V^*) \to \mathfrak{Ug}$ is a $\mathfrak{g}$-module map, and satisfies the "Jacobi" identity.

Verifying the Jacobi identity is easy, since in computing all relevant iterated commutators, the inner one always yields a scalar here. To check that $[-,-]$ is a $\mathfrak{g}$-module map, we compute:

$$X([v,v^*]) = [X(v), v^*] + [v, X(v^*)] = \beta_0 v^*(X(v)) + \beta_0 X(v^*)(v) = 0,$$

by contragredience of V and V^* (here, $v \in V, v^* \in V^*$). The other side has $X(\beta_0 v^*(v))$, which vanishes since k is a trivial $\mathfrak{g}$-module. The other verifications are trivial; hence H_{β_0} is a flat deformation.

To check (HRTA2) (and the rest of (HRTA1), including defining $H, B_\pm$), we refer to Proposition 10.2 below. Set $M = \mathfrak{n}_- \oplus \mathfrak{n}_+ \oplus V \oplus V^*$; we now show that the second characterization holds here. Set $d = 1 + \dim \mathfrak{h}$, and $K = (\mathfrak{h} \oplus k\tau)^\perp \subset \mathfrak{h}_0^* \subset \widetilde{\mathfrak{h}}^*$. Then the fundamental weights ω_i and the weight $\lambda_0 \in \mathfrak{h}_0^*$ give a basis, in terms of which, $\Pi(\mathfrak{n}_- \oplus \mathfrak{n}_+ \oplus V \oplus V^*) \subset \mathbb{Q}^d \setminus 0$.

Now by Proposition 10.2, there exists $\delta \in \widetilde{\mathfrak{h}}$ such that $\Pi(M)(\delta) \subset \mathbb{Z} \setminus 0$. It is possible to choose δ such that $\mathfrak{g} \oplus V \oplus V^* = (V^* \oplus \mathfrak{n}_-) \oplus (\mathfrak{h} \oplus \mathfrak{h}_0) \oplus (V \oplus \mathfrak{n}_+)$, where $\mathfrak{b}_+ := \mathfrak{n}_+ \ltimes V, \mathfrak{b}_- := \mathfrak{n}_- \ltimes V^*$ are the spans of all δ-eigenvectors in $\mathfrak{g} \oplus V \oplus V^*$ with eigenvalues in $\mathbb{N}, -\mathbb{N}$ respectively. (This is by choice for V, and since $\Pi(V^*) = -\Pi(V)$ by contragredience.) Now write $B_\pm = \mathfrak{U}\mathfrak{b}_\pm$; the rest of (HRTA2) is then proved similar to the case of an RTLA.

(2) Note that $\sum_i v_i v_i^*$ commutes with $\mathfrak{g}$ by Proposition 3.9 (for $H = \mathfrak{Ug}$, where $X \in \mathfrak{g}$ acts on H_{β_0} by $X(a) := [X,a]$), and so does $\tau \in \mathfrak{h}_0$, where $\lambda_0(\tau) = 1$. Thus, it remains to show that V, V^* commute with this element. But we now compute:

$$\left[v_j, \sum_i v_i v_i^* + \beta_0 \tau \right] = [v_j, v_j v_j^*] + \beta_0 [v_j, \tau] = v_j \cdot \beta_0 - \beta_0 v_j = 0,$$

and similarly for $[v_j^*, \sum_i v_i v_i^* + \beta_0 \tau]$.

$\square$

In the above proof, we used $\mathfrak{h} = k \cdot h_0$ to be one-dimensional (and $H_0 = \mathfrak{U}\mathfrak{h}_0$). This is not something remarkable at all, as is shown by the following characterization of (HRTA2) for infinitesimal Hecke algebras.

Proposition 10.2. *Suppose a k-algebra A is generated by an abelian Lie algebra $\mathfrak{h}$ and a finite-dimensional semisimple $\mathfrak{h}$-module M, with $M_0 = 0$. The following are equivalent:*

(1) *"HRTA2" holds; in other words, there exist*
- *a Lie subalgebra $\mathfrak{h}_0 \subset \mathfrak{h}$,*
- *a decomposition $M = M_+ \oplus M_-$ into $\mathfrak{h}$-submodules, and*
- *a linearly independent set $\Delta \subset \mathfrak{h}_0^*$,*

such that $M_\pm = \bigoplus_{\mu \in \pm \mathbb{Z}_{\geqslant 0}\Delta} (M_\pm)_\mu$.

(2) *There exists a codimension d subspace $K \subset \mathfrak{h}^*$ (for some d), such that modulo K, and up to a change of basis, $\overline{\Pi(M)} \subset \mathbb{Q}^d \setminus 0$.*

(3) *There exists $\delta \in \mathfrak{h}$ such that $\Pi(M)(\delta) \subset \mathbb{Z} \setminus 0$.*

Remark 10.3.

(1) For example, for the infinitesimal Hecke algebras associated to $(\mathfrak{gl}_n, k^n \oplus (k^n)^*)$ and $(\mathfrak{sp}_{2n}, k^{2n})$ (which were characterized in [EGG]), the second condition is easily verified, for $M = V \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, $K = 0$, and the basis consisting of the fundamental weights (and also one more weight in $\mathfrak{h}_0^*$ for $\mathfrak{gl}_n$).

(2) The first condition is what is needed to show that A is an HRTA; the second is what typically comes as "given data" for A; and the third is needed to apply Ginzburg's Generalized Duflo Theorem.

(3) Next - the conditions in (HRTA2) are stated in terms of $B_\pm$, unlike the first statement above. However, in the case of infinitesimal Hecke algebras H_β, the spaces $M_\pm$ are typically Lie algebras if $\beta = 0$, and $B_\pm$, which are the subalgebras generated by $M_\pm$ inside H_β, are deformations of $\mathfrak{U}(M_\pm) \subset H_0$. In particular, given (HRTA1), some PBW property yields the regularity conditions inside (HRTA2).

Proof. We prove a series of cyclic implications.

(1) $\Rightarrow$ (2): Define $d = |\Delta|$, $K := \mathfrak{h}_0^\perp \subset \mathfrak{h}^*$, and the basis in $\mathfrak{h}^*/K = \mathfrak{h}_0^*$ to be Δ. Then (2) follows.

(2) $\Rightarrow$ (3): Since $\mathbb{Q}$ is an infinite field, and $0 \notin \overline{\Pi(M)}$, choose a hyperplane $K_1 \subset \mathbb{Q}^d \setminus \overline{\Pi(M)}$, and consider $0 \neq h_0 \in (K_1 + K)^\perp = (\overline{K_1})^\perp$. Since these weights are all in a $\mathbb{Q}$-vector space, there exists $c \in k^\times$, such that

$$\alpha(h_0) \in \mathbb{Q}^\times \cdot c \ \forall \alpha \in \Pi(M) \subset \mathfrak{h}^*.$$

Rescale h_0, using that $\operatorname{char}(k) = 0$, to get δ such that $\alpha(\delta) \in \pm\mathbb{N} \ \forall \alpha \in \Pi(M)$.

(3) $\Rightarrow$ (1): Set $\mathfrak{h}_0 = k \cdot \delta$, $M_\pm := \bigoplus_{n \in \pm\mathbb{N}} M_n$ with respect to $\operatorname{ad}\delta$, and $\alpha \in \mathfrak{h}_0^*$ via: $\alpha(\delta) = 1$. $\square$

10.2. **The general linear case.** We now show that all infinitesimal Hecke algebras $H_\beta(\mathfrak{gl}_n)$ are Hopf RTAs (i.e., for all n, β). Let us recall the definition of these algebras from [EGG, §4.1.1.] first:

Set $\mathfrak{g} = \mathfrak{gl}_n$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = k^n$ and $\mathfrak{h}^*$ is its dual representation. We (again) identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the trace pairing $\mathfrak{g} \times \mathfrak{g} \to k : \ (A, B) \mapsto \operatorname{tr}(AB)$, and identify $\mathfrak{U}\mathfrak{g}$ with $\operatorname{Sym}\mathfrak{g}$ via the symmetrization map.

Then for any $x \in \mathfrak{h}^*, \ y \in \mathfrak{h}, \ A \in \mathfrak{g}$, one writes

$$(x, (1 - TA)^{-1} y) \det(1 - TA)^{-1} = r_0(x, y)(A) + r_1(x, y)(A)T + \dots$$

where $r_i(x, y)$ is a polynomial function on $\mathfrak{g}$, for all i.

Now for each polynomial $\beta = \beta_0 + \beta_1 T + \beta_2 T^2 + \dots \in k[T]$, the authors define in [EGG] the algebra H_β as a quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathfrak{U}\mathfrak{g}$ by the relations

$$[x, x'] = 0, \qquad [y, y'] = 0, \qquad [y, x] = \beta_0 r_0(x, y) + \beta_1 r_1(x, y) + \dots$$

for all $x, x' \in \mathfrak{h}^*, \ y, y' \in \mathfrak{h}$. It is proved in [EGG] that these algebras are infinitesimal Hecke algebras (so the "PBW property" holds). Also note that if $\beta \equiv 0$, then $H_0 = \mathfrak{U}(\mathfrak{gl}_n \ltimes (\mathfrak{h} \oplus \mathfrak{h}^*))$.

Proposition 10.4. *$H_\beta(\mathfrak{gl}_n)$ is a Hopf RTA for every n, β.*

Proof. We first make the necessary identifications: set $\mathfrak{g}_0 = \mathfrak{sl}_n, \mathfrak{h}_0 = k \cdot \mathrm{id}_n \subset \mathfrak{g} = \mathfrak{gl}_n$, and $V = k^n$ (so $\lambda_0(\mathrm{id}_n) = 1$). Then this algebra satisfies (HRTA1) by [EGG], where $B_\pm \cong \mathfrak{U}(\mathfrak{n}_+ \ltimes V)$ and $\mathfrak{U}(\mathfrak{n}_- \ltimes V^*)$ respectively. Moreover, the verification of (HRTA2) is the same as what is done in proving Propositions 10.1 and 10.2.

We now present a map from [KT], that we claim is an anti-involution for general n, β: j takes $X \in \mathfrak{gl}_n$ to X^T, and $v_i \leftrightarrow -v_i^* \ \forall i$.

We have to verify three kinds of relations; two of these are not hard. First, that j is an anti-involution on $\mathfrak{gl}_n$ is clear, since $j(X) = X^T$ is anti-multiplicative. Next, $[e_{ij}, v_k] = \delta_{jk} v_i$ and $[e_{ji}, v_k^*] = -\delta_{jk} v_i^*$ are clearly interchanged by j, so these relations are also preserved. Third, $[v_1, v_2] \equiv [v_1^*, v_2^*] \equiv 0$ are also j-stable relations (for $v_i \in \mathfrak{h}, \ v_i^* \in \mathfrak{h}^*$).

It remains to consider the relations: $[v_k^*, v_l] = \sum_{i \geq 0} \beta_i r_i(v_l, v_k^*)$. Note that each $r_i(v, v^*)$ is in $\mathfrak{U}\mathfrak{g}$ - and at the same time, identified with a function $r_i(v, v^*)(-) : \mathfrak{g} \to k$, via the symmetrization map.

Let us first analyze the left side: $j([v_k^*, v_l]) = [v_l^*, v_k] = \sum_{i \geq 0} \beta_i r_i(v_k, v_l^*)$. Now recall how the r_k's were defined. Treating $v \in \mathfrak{h}$ and $v^* \in \mathfrak{h}^*$ as column and row vectors respectively, any inner product (v^*, Av) now merely

is matrix multiplication v^*Av. Thus, we compute (inside our algebra):

$$\sum_{i\geq 0} r_i(v_k, v_l^*)(A)T^i = (v_l^*, (1-TA)^{-1}v_k)\det(1-TA)^{-1}$$
$$= v_l^T(1-TA)^{-1}v_k \cdot \det(1-TA)^{-1} = v_k^T(1-TA^T)^{-1}v_l \cdot \det(1-TA^T)^{-1}$$
$$= (v_k^*, (1-TA^T)^{-1}v_l)\det(1-TA^T)^{-1} = \sum_{i\geq 0} r_i(v_l, v_k^*)(A^T)T^i.$$

We claim that $j(r_i(v_l, v_k^*)(A)) = r_i(v_l, v_k^*)(A^T)$ for all i, k, l. But using the above computation, we would then get:

$$\begin{aligned} j\left(\sum_{i\geq 0} \beta_i r_i(v_l, v_k^*)(A)\right) &= \sum_{i\geq 0} \beta_i r_i(v_l, v_k^*)(A^T) = \sum_{i\geq 0} \beta_i r_i(v_k, v_l^*)(A) \\ &= [v_l^*, v_k] = j([v_k^*, v_l]), \end{aligned}$$

which will show that the j does indeed preserve these last relations. We are now done, since our claim follows from Proposition 10.12. $\square$

Remark 10.5. This HRTA structure is not unique; for instance, one checks that taking δ to be the diagonal matrix with entries $(2n-1, 2n-5, \dots, 3-2n)$ works for $H_\beta(\mathfrak{gl}_n, k^n \oplus (k^n)^*)$ for all n and all linear $\beta = \beta_0 + \beta_1 T$.

Finally, we note that by the above result, one can define $\mathcal{O}$ over $H_\beta(\mathfrak{gl}_n)$ for all β, and one now has:

Theorem 10.6. *For all β, the category $\mathcal{O}$ over $H_\beta(\mathfrak{gl}_2)$ splits into a direct sum of highest weight categories.*

This is because by [Tik], we know the center of this algebra, and explicit calculations prove that it satisfies Condition (S4).

10.3. **The symplectic case.** These algebras are generated by $\mathfrak{g} = \mathfrak{sp}(2n)$ and its natural representation, $V = k^{2n}$. The bases for these that we use are e_i, e_{i+n} for k^{2n} (with $1 \leq i \leq n$), and (e.g., see [KT])

$$u_{jk} := e_{jk} - e_{k+n,j+n}, \quad v_{jk} := e_{j,k+n} + e_{k,j+n}, \quad w_{jk} := e_{j+n,k} + e_{k+n,j}.$$

As mentioned in [KT], given a scalar parameter β_0, these algebras H_{β_0} are generated by $\mathfrak{sp}(2n) \oplus V$, modulo the usual Lie algebra relations for $\mathfrak{sp}(2n)$, the "semidirect product" relations $[X, v] = X(v)$ for all $X \in \mathfrak{g}, v \in V$, and the relations $[e_i, e_j] = \beta_0 \delta_{|i-j|,n}(i-j)/n$.

Proposition 10.7. *The algebras H_{β_0} are Hopf RTAs.*

There are other Hopf RTAs, e.g., symplectic oscillator algebras $H_\beta(\mathfrak{sl}_2, k^2)$ for any β. Moreover, for all n and all β, we show below that there always exists the anti-involution of (HRTA3).

Proof. Define $h_0 := \operatorname{diag}(n, n-1, \dots, 1, -n, -(n-1), \dots, -1)$, and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, its standard triangular decomposition. Then $V_n := \mathfrak{g} \oplus V$ has a basis of eigenvectors for $\mathfrak{h}$, and in particular, for h_0 (with eigenvalues in

$\mathbb{Z}$). Write $V_n = \mathfrak{n}'_- \oplus \mathfrak{h}' \oplus \mathfrak{n}'_+$, a decomposition into spans of eigenvectors with negative, zero, and positive eigenvalues respectively. Then $\mathfrak{h}'$ is indeed the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and $\mathfrak{n}'_\pm = \mathfrak{n}_\pm \oplus \mathfrak{h}_\pm$ are Lie subalgebras in H_β, where $\mathfrak{h}_\pm$ are the spans of $\{e_1, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{2n}\}$ respectively.

Next, define $\mathfrak{h}_0 := kh_0,\ H = \operatorname{Sym}\mathfrak{h},\ H_0 = \operatorname{Sym}\mathfrak{h}_0,\ B_\pm = \mathfrak{U}\mathfrak{n}'_\pm$. Then H_β has the required triangular decomposition, by [EGG], and H is a commutative Hopf algebra with sub-Hopf algebra H_0.

Moreover, $G = \mathfrak{h}^*$ surjects onto $G_0 = \mathfrak{h}_0^* \cong k$. Define $\mathcal{P}_0 = \mathbb{Z}$ (which is generated by the $\operatorname{ad} h_0$-weights of V_n). The remaining part of axiom (HRTA2) is shown as for RTLAs. Finally, there exists an anti-involution as desired - this was shown in [KT]:

$$j : u_{jk} \leftrightarrow u_{kj},\ v_{jk} \leftrightarrow -w_{jk},\ e_i \leftrightarrow e_{i+n}. \tag{10.8}$$

□

Next, we show that all H_β possess an anti-involution as in (HRTA3). Let us first define $H_\beta = H_\beta(\mathfrak{sp}_{2n}, k^{2n})$ for general n, β. Denote by ω the symplectic form on V; one then identifies $\mathfrak{g}$ with $\mathfrak{g}^*$ via the pairing $\mathfrak{g} \times \mathfrak{g} \to k,\ (A, B) \mapsto \operatorname{tr}(AB)$, and $\operatorname{Sym}\mathfrak{g}$ with $\mathfrak{U}\mathfrak{g}$ via the symmetrization map. Write

$$\omega(x, (1 - T^2A^2)^{-1}y) \det(1 - TA)^{-1} = l_0(x, y)(A) + l_2(x, y)(A)T^2 + \dots$$

where $x, y \in V, A \in \mathfrak{g}$, and $\forall i, l_i(x, y) \in \operatorname{Sym}\mathfrak{g} \cong \mathfrak{U}\mathfrak{g}$ is a polynomial in $\mathfrak{g}$.

For each polynomial $\beta = \beta_0 + \beta_2T^2 + \beta_4T^4 + \dots \in k[T]$, in [EGG] the authors define the algebra H_β to be the quotient of $TV \rtimes \mathfrak{U}\mathfrak{g}$ by the relations

$$[x, y] = \beta_0 l_0(x, y) + \beta_2 l_2(x, y) + \dots$$

for all $x, y \in V$. Our main result here is:

Proposition 10.9. *For all n, β, the map $j : H_\beta \to H_\beta$, defined as in equation* (10.8)*, is an anti-involution that fixes $H = \operatorname{Sym}\mathfrak{h}$. Moreover, the conditions of Proposition 10.2 are true.*

To show this result, we need the following preliminary lemma, whose proof is by straightforward computations.

Lemma 10.10.

1. *The map j on $\mathfrak{sp}_{2n}$ can be extended to all of $\mathfrak{gl}_{2n}$, via: $j(C) = \tau C^T \tau$ - where $\tau = \tau^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2n)$.*
2. *One has $\omega(x, Cy) = \omega(j(y), j(C)j(x))$, for all $x, y \in k^{2n},\ C \in \mathfrak{gl}_{2n}$ (using j as in the previous part).*
3. $j\left((1 - T^2A^2)^{-1}\right) = (1 - T^2 j(A)^2)^{-1}$.

Proof of Proposition 10.9. We can now finish the proof of the result. That the conditions of Proposition 10.2 hold here, follows from defining $\delta := h_0$, the special element from the proof of Proposition 10.7. As for the proposed anti-involution, it is not hard to check that j is an anti-involution on $\mathfrak{sp}_{2n}$,

that preserves the relations $[X, v] = X(v)$ for $X \in \mathfrak{sp}_{2n}$ and $v \in k^{2n}$. We are left to consider the relations $[x, y]$. We compute (using the above lemma):

$$\begin{aligned}
\sum_{i \geq 0} l_{2i}(x,y)(A)T^{2i} &= \omega(x, (1 - T^2A^2)^{-1}y) \det(1 - TA)^{-1} \\
&= \omega(j(y), (1 - T^2 j(A)^2)^{-1} j(x)) \det(1 - Tj(A))^{-1} \\
&= \sum_{i \geq 0} l_{2i}(j(y), j(x))(j(A))T^{2i},
\end{aligned}$$

where the second equality is not hard to show. In particular, replacing A by $j(A)$ and equating coefficients of T, we get:

$$l_{2i}(x,y)(j(A)) = l_{2i}(j(y), j(x))(A) \ \forall x, y \in k^{2n}, i \geq 0. \tag{10.11}$$

We can now compute:

$$\begin{aligned}
j([x,y]) &= j\left(\sum \beta_i l_{2i}(x,y)(A)\right) = \sum \beta_i l_{2i}(x,y)(j(A)) \\
&= \sum_i \beta_i l_{2i}(j(y), j(x))(A) = [j(y), j(x)],
\end{aligned}$$

where the first and last equalities are by definition, the second uses Proposition 10.12 (via the trace form), and the third follows from equation (10.11). □

10.4. **The symmetrization map and anti-involutions.** We finally mention a result that was used in proving that every infinitesimal Hecke algebras over $\mathfrak{gl}_n$ has an anti-involution needed to make it an Hopf RTA.

Proposition 10.12. *Suppose $\mathfrak{g}$ is any Lie algebra, and we identify* Sym $\mathfrak{g}$ *with $\mathfrak{U}\mathfrak{g}$ via the* symmetrization map

$$\mathrm{sym} : X_1 \dots X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \dots X_{\sigma(n)}.$$

Suppose j is a Lie algebra anti-involution of $\mathfrak{g}$. Then the automorphism j of Sym $\mathfrak{g}$ *is transferred to j via* sym.

Applying this to infinitesimal Hecke algebras over $\mathfrak{gl}_n$ in the proof of Proposition 10.4, we get (via a further identification of $\mathfrak{g} \leftrightarrow \mathfrak{g}^*$ by the trace form):

$$r_i(v, v^*)(A^T) = r_i(v, v^*)(j(A)) = j(r_i(v, v^*)(A)),$$

as desired. A similar application yields the anti-involution mentioned above for infinitesimal Hecke algebras over $\mathfrak{sp}_{2n}$.

Proof. We easily check that we have the following commuting diagram:

$$\begin{array}{ccc}
\mathrm{Sym}\,\mathfrak{g} & \xrightarrow{j} & \mathrm{Sym}\,\mathfrak{g} \\
{\scriptstyle \mathrm{sym}}\downarrow & & \downarrow{\scriptstyle \mathrm{sym}} \\
\mathfrak{U}\mathfrak{g} & \xrightarrow{j} & \mathfrak{U}\mathfrak{g}
\end{array}$$

and all maps are vector space isomorphisms. □

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